PHYS260 - 0105

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Contents

Overview, Charges, and Forces

Announcements

- Homework 1 is due 11:59 pm, on this Thursday, 8/29
- Questions will be asked about the syllabus on the exam
- Quiz 1 is due at 11:59 pm on Tuesday, 9/3. Open book, 20 minutes. (mostly conceptual questions)

Overview

About electricity, magnetism and thermodynamics.

Lecture notes will be posted on ELMS before lectures.

Note-taking is encouraged to help your learning.

Exams closely follow the lectures, not necessarily the textbook.

This is a fast-paced course. Ask for help early.

Contact the professor via ELMS. (Refer to her as Prof. Girvan).

Office hours: Monday, 1-2 pm and Tuesday, 3:30-4:30 pm, or by appointment.

Homework will usually be due weekly via Expert TA. There may be different due dates for 010x and 030x sections. Links to the assignments will also be accessible via ELMS.

One of the main ways you can understand physics is by doing the homework. You should focus on being able to do the homework on your own.

You will have 2 closed-book midterm exams (with a letter single-sided formula sheet.) The higher score will be 22% of your final grade, and your lower score will count as 15% of your final grade.

Slido will be used instead of PointSolutions for polling/clicker questions. They are ungraded.

Don't cheat.

Practice exams will be given out with a worked-out answer key.

There are lots of office hours available (see slides/ELMS).

Chapter 5: Electric Charges and Fields

Electic Charge

Electric phenomena depend on charges

- There are two kinds of charge, positive and negative.

- Electrons and protons, parts of atoms, are the basic charges of matter
- Usually, charges come from the transfer of electrons

How they behave:
- Two charges of the same type repel; opposite types attract
- charge can be transferred
- charge is conserved

Conductors and insulators
- Conductors are materials where charge moves easily
- Insulators are materials on which charge is immobile

Coulomb's law
Like gravity, it's an inverse square law.

$$|F| = \frac{k|Q_1||Q_2|}{r^2}$$

Experiment
Two untouched glass rods, brought together, do nothing.

What happens when both are rubbed with silk? Perhaps almost nothing, but they're supposed to repel each other because they have the same charge.

Plastic rubbed with wool and glass rubbed with silk will attract each other because they have opposite charges.

The strength of the force between charged objects depends inversely on the square of the distance (closer together objects have more of an effect).

Q1
What will happen if you put a neutral object near a charged object?

The neutral object will be attracted to the charged object.

This is because the charges in the neutral object will move around to have opposite charges closer to the charged object.

Electric Charge and the structure of matter
The particles of the atom are the negative electrons, the positive protons, and the neutral neutrons.

Protons and neutrons are in the nucleus.

Neutral atoms have the same number of protons and electrons.

Positive ions have electrons removed, and negative ions have excess electrons.

Conservation of charge
- The proton and the electron have the same magnitude charge.
- All charge is quantized into these units of charge
- The sum of all electric charges in a closed system is constant.

The SI union of charge is the coulomb:

$$e = 1.6 \times 10^{-19} \text{ C}$$

Insulators and Conductors

Charge spreads out on the surface of conductors but remains in a fixed location in/on insulators.

Charging and Discharging

If objects have different charges and they touch, they will exchange charges until they have the same charge.

Grounding

Grounding removes excess charge by connecting charge to some object of large size.

This large object is called a ground and is seen as an infinite reservoir of electrons.

Charge Polarization

A charged rod held close to an electroscope will cause the leaves to repel each other by moving charges toward the top of the electroscope, leaving similar charges on the bottom, causing the leaves also on the bottom to repel each other.

When an object has this directional splitting of charge, it is called polarized. Charge polarization is a slight separation of a neutral object's positive and negative charges.

When the force causing this to happen leaves, it quickly returns to normal.

Charging by induction

You can use polarization to transfer charge.

1. Object B touches object A on top
2. Polarize the object A-object B system with a positively charged rod from above
3. Move Object B away from object A and the charged rod
4. Stop polarizing object A

The electric dipole

An electric dipole is a system of two charges with equal magnitude but opposite signs, separated by a small distance.

When an insulator is brought near an external charge, all the individual atoms inside the insulator become polarized. The many polarized atoms create a net polarization force, even though the electrons can't move.

Coulomb's Law

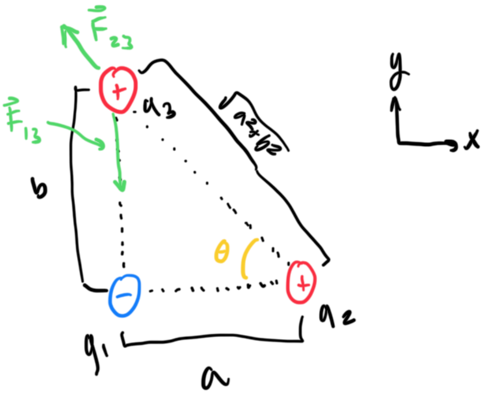
The magnitude of the electric force between two point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them.

$$|F| = \frac{k|Q_1||Q_2|}{r^2}$$

$k = 8.99 \times 10^9 \text{ N m }^2/\text{ C}^2$

Something to note: this is applied to point charges, not to whole objects.

Example



$$\vec{F}_{13} = \frac{k|q_1 \cdot q_3|}{b^2} \cdot (-\hat{j})$$

down

$$\vec{F}_{23} = \frac{k|q_2 \cdot q_3|}{a^2+b^2} \cdot \left(-\hat{i}\cos\theta + \hat{j}\sin\theta\right)$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \vec{F}_{23} = \frac{k|q_2 q_3|}{(a^2+b^2)^{3/2}}\left(-a\hat{i} + b\hat{j}\right)$$

$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23} = \ldots = (4.83\hat{i} + 3.84\hat{j}) \times 10^{4} \text{ N}$$

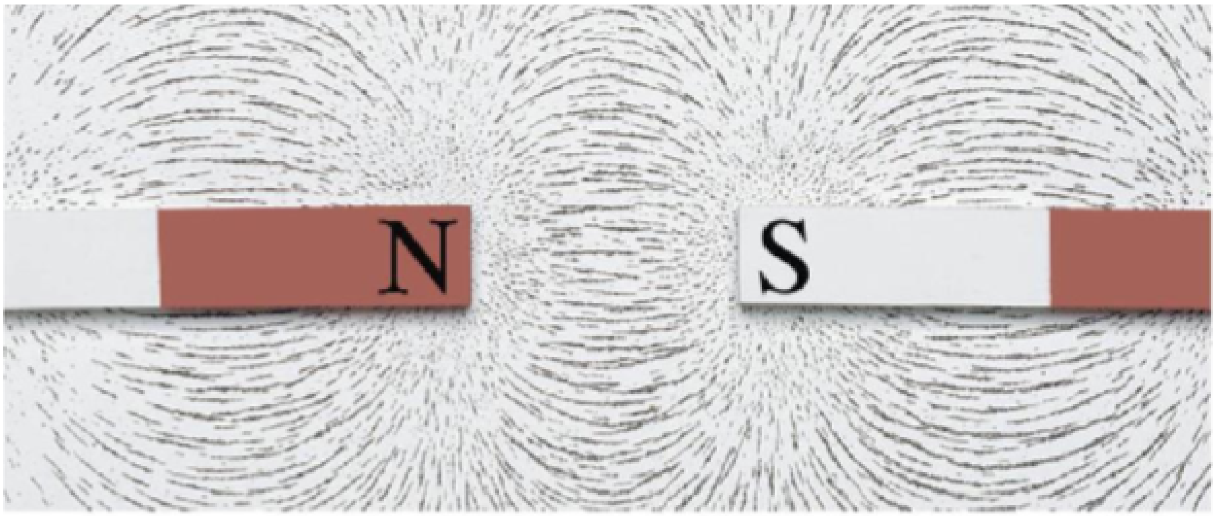
Notation

$$\vec{F}_{12} = \text{Force by 1 on 2} \; \vec{r}_{12} = \text{Vector from 1 to 2}$$

Summary: Charge Basics

- Two kinds of charge: positive and negative
- Like charges repel, opposite charges attract.
- Neutral objects have an equal mixture of positive and negative charge
- Charge is quantized; it comes in multiples of the value of an electron
- Materials can be separated into two types: conductors, where charge can move, and insulators, where charge stays stationary
- the SI unit of the charge is the Coulomb, and $e = 1.6 * 10^{-19}$ C

The Field Model



The photo shows the patterns of iron filings formed when they are around a magnet.

This suggests that magnets create fields. This field is called a magnetic field. We will study the related electric field in this chapter.

The field model states that charges interact via the electric field:

- The electric field exerts electric forces on charges
- Source charges create the field
- The field is composed of vectors at every point in space
- A charge does not feel its own field.

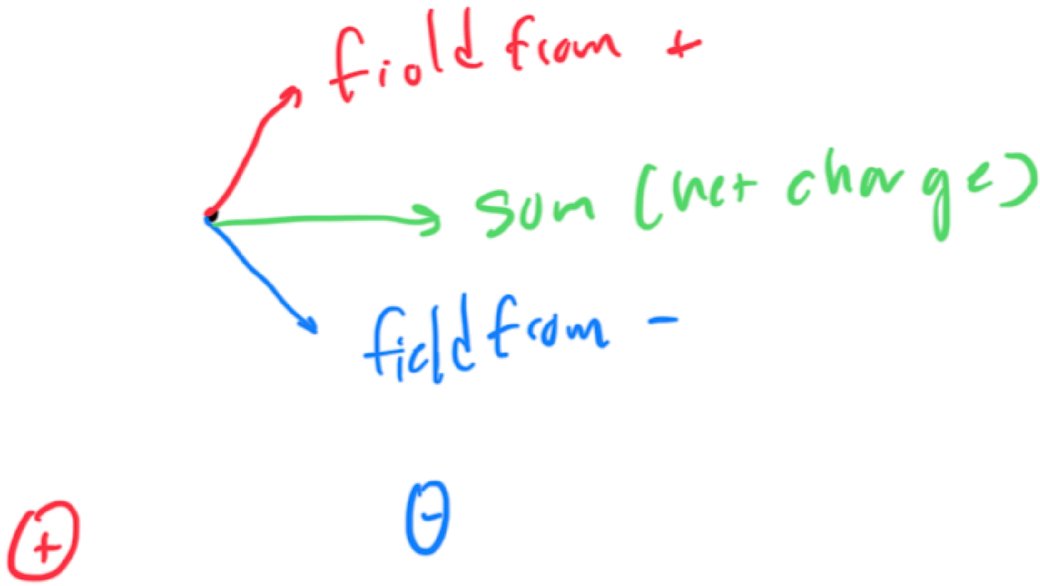
Point Charge Example

For positive source charges, the electric charge formed goes outward, and for negative charges, it goes inward.

$$\vec{E} = k\frac{Q}{r^2}\hat{r}$$

Then, the force on a charge q will be $\vec{F} = q\vec{E}$.

Electric fields make things simpler because the total electric field is just the sum of all electric fields caused by source charges.



Continuous Charge Distributions

What if the charge is continuous?

For macroscopic charged objects, like rods or disks, we can think of the charge as having a continuous distribution.

A charged object is characterized by its charge density, the charge per length, area, or volume.

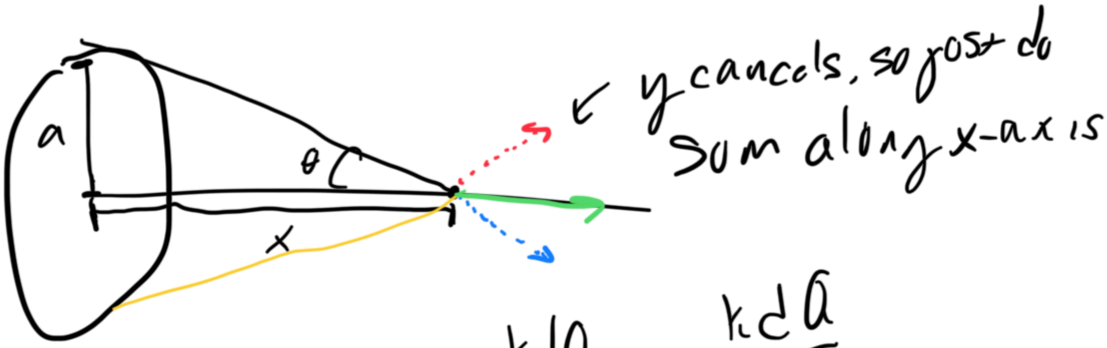
Then, you can sum the entire electric field with an integral.

Example: A ring of charge

A thin, ring-shaped object of radius a holds a total charge $+Q$ distributed uniformly around it. Determine the electric field at a point P on its axis, at a distance x from the center.

We can see that the energy is only along one axis, so we can ignore everything else.

We also know the total charge is Q.



$$dE = |d\vec{E}| = \frac{kdQ}{r^2} = \frac{kdQ}{a^2+x^2}$$

$$dE_x = dE \cdot \cos\theta = \frac{kxdQ}{(a^2+x^2)^{3/2}}$$

All segments have the same dE_x

$$E_x = \int dE_x = \frac{kx}{(x^2+a^2)^{3/2}} \cdot \int dQ = \frac{kxQ}{(x^2+a^2)^{3/2}}$$

The answer is

$$\vec{E} = \frac{kQx}{(a^2+x^2)^{\frac{3}{2}}}$$

What if $x \gg a$?

$$\vec{E} \approx \frac{kQx}{(x^2)^{\frac{3}{2}}}\hat{i} = \frac{kQx}{x^3} = \frac{kQ}{x^2}\hat{i}$$

It basically becomes a point charge!

With enough distance, anything becomes a point charge.

Btw whoever is reading this, I fucked up an example here (not pictured), see the slides for the answer. I will probably do the example again myself but correctly.

Motion of a Charged particle in an Electric Field

The electric field exerts a force on a charged particle: $\vec{F} = q\vec{E}$.

This may cause the particle to accelerate.

If there are no other forces, this $\vec{a} = \frac{q}{m}\vec{E}$.

In a uniform field, the acceleration is constant. $a = q\frac{E}{m}$.

For example, if there is an electron moving to the right, you need a force to the left to stop it. Because it is an electron, the field must be to the right to stop the electron.

Electric Field Lines

Electric Field lines are continuous curves tangent to the electric field vectors.

Closely spaced field lines indicate a greater field strength.

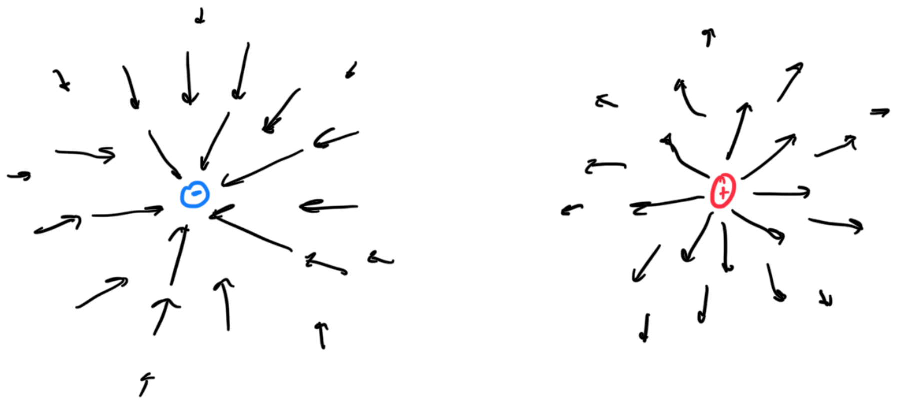
Electric field lines never cross, and go from positive charges to negative charges.

Electric Field

$$\vec{E} = k\frac{Q}{r^2}\hat{r}$$

$\hat{r}$ is a unit vector that goes radially outwards from the charge, but often it's easier to add the direction afterwards, and just use the formula for the magnitude.

This formula then means that the direction of the field is radially outward for positive charges and radially inward for negative charges:



The force on a charge of charge q is then $\vec{F} = q\vec{E}$

Using Calculus

If we have something that isn't a point charge, we can divide the total charge Q into many smaller charges $\mathrm{d}Q$. Then, $\vec{E} = \int \frac{k\hat{r}}{r^2}\,\mathrm{d}Q$.

Often we will rewrite $\mathrm{d}Q$ into a factor of something more relevant like $\mathrm{d}x$. For example, $\mathrm{d}Q = Q\frac{\mathrm{d}x}{L}$

Gauss's Law

- Given a distribution of charge, we can enclose the charge in a surface.
- Given the distrubtion of the charge, we can find the distrubtion of charge on the surface.
- With that distrubtion of the surface, we can find the total charge enclosed within the surface.

Examples

If we have a box with no charge inside of it, and the electric field is zero everywhere, the flux is zero.

If we have negative charges inside the box, there will be inward pointing electric flux on the surface.

If we have positive charges inside the box, there will be outward pointing electric flux on the surface.

If there are both positive and negative charges in the box, the net flux is zero, but its inward near the negative charge and outward near the positive charge.

If there is a positive charged surface nearby, there is flux on the surface, but the net charge is zero.

The electric flux is linear with the amount of enclosed charge, and independent of the size of the box.

Calculation

$$\Phi_e = \vec{E} \cdot \vec{A}$$

The dot is for the dot product.

$$\vec{A} = A\hat{n}$$

A is the area of the surface, and $\hat{n}$ is vector normal to the surface.

Generally, this can be expressed as an integral for a surface S:

$$\Phi_e = \int_S \vec{E} \cdot \mathrm{d}\vec{A}$$

Special Cases

If $\vec{E}$ is always tangent to the surface, the flux is zero.

IF $\vec{E}$ is always perpendicular to the surface and has the same magitude E everywhere on the surface, the flux is EA.

$$\Phi_e = \int_S \vec{E} \cdot \mathrm{d}\vec{A} = \int E_\perp \,\mathrm{d}A = \int E \cos(\phi) \,\mathrm{d}A$$

$E_\perp$ is the perpendicular part of the electric field $\vec{E}$. ϕ is the angle with the normal vector.

Example: Sphere

Consider the flux through a sphere S of radius r around a point charge of charge q:

$$\begin{aligned}\Phi_e &= \oint_S \vec{E} \cdot \mathrm{d}\vec{A} \\ &= EA_{\text{sphere}}\end{aligned}$$

We can say the above because of the special case with perpendicular to the surface.

$$k = \frac{1}{4\pi\varepsilon_0}$$

$$= \frac{q}{(4\pi\varepsilon_0)r^2}(4\pi r^2)$$
$$= \frac{q}{\varepsilon_0}$$

We can see how the flux is independent of the size of the sphere.

Generalize

The electric flux through any arbitrary closed surface surrounding a point charge q may be broken up into spherical and radial pieces (calculus-style), and since the flux is size independent, you can use multiple without a problem.

Therefore, the total flux through the surface is the same as above:

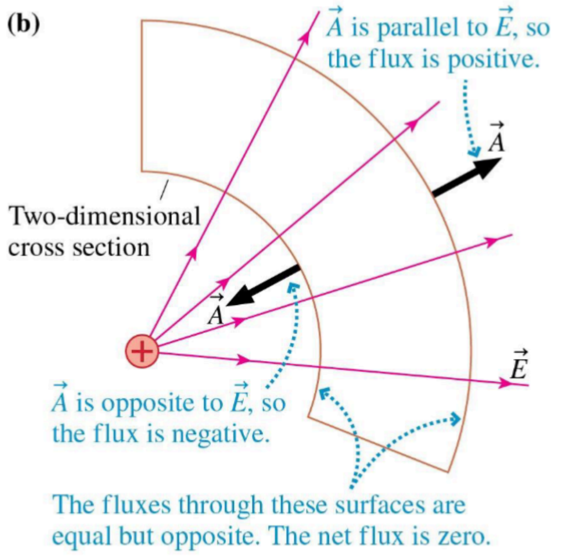
$$\Phi_e = \frac{q}{\varepsilon_0}$$

The Formula

$$\Phi_e = \oint \vec{E}\,\mathrm{d}\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Where Q_{enclosed} is the total charge enclosed by the surface.

This can be explained by having two spheres surrounding an outsider charge:



Exploiting Symmetries

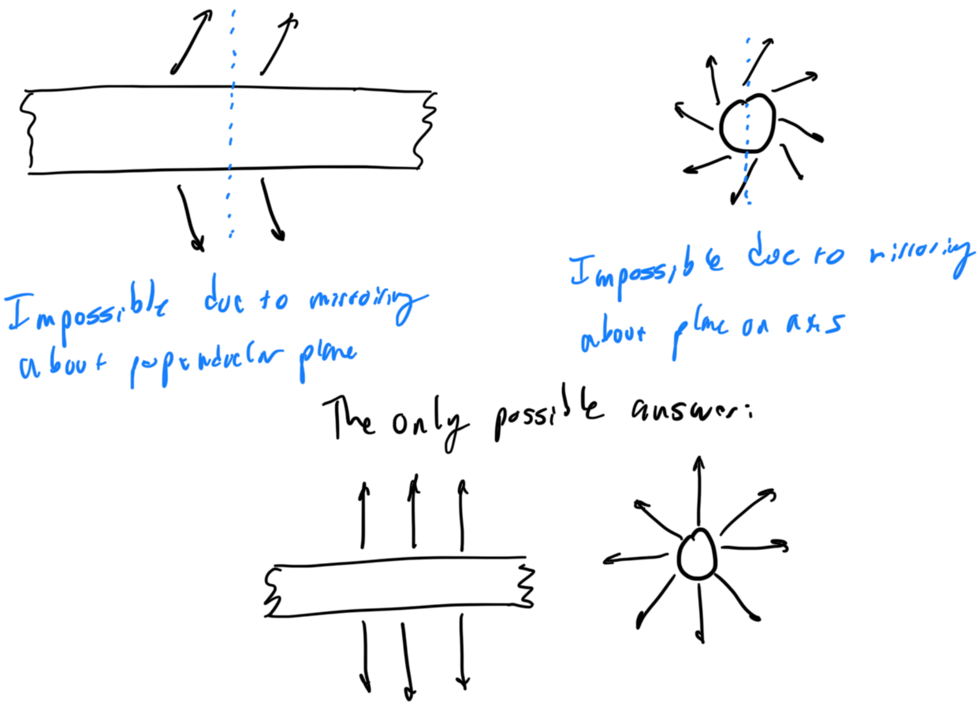
Infinite Cylinder

Imagine you have a infinitely long, charged cylinder. What is the electric field?

What are the symmetries?

- Translating the rod in the direction the cylinder axis does nothing (it's infinitely long!)
- Rotating the cylinder about the axis
- Mirroring along any plane perpendicular to the axis
- Mirroring along any plane on the axis

Therefore:

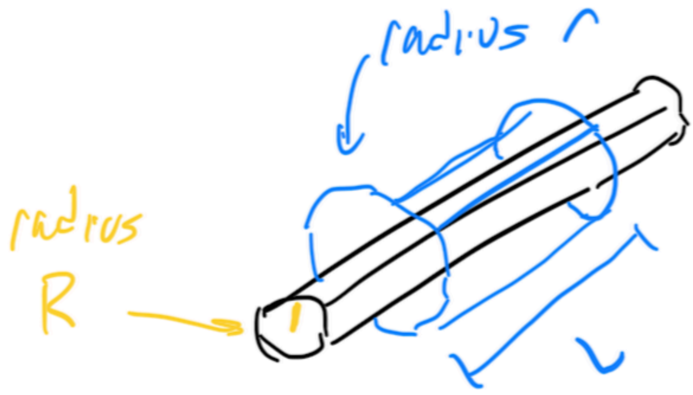


Use Gauss's law to find the electric field inside and outisde an infinitely long, uniformly charged cylinder with radius R and charge density ρ.

Find $E(r)$, where r is the distance from the axis.

Case 1: $r > R$

Construct the Gaussian surface S around the cylinder with radius r and length L.



We then can do the surface integral:

$$\Phi_e = \oint_S \vec{E} \cdot \mathrm{d}\vec{E}$$

$$
\begin{aligned}
&= \Phi_{\text{ends}} + \quad \Phi_{\text{sides}} \\
&= 0 + \qquad EA_{\text{sides}}
\end{aligned}
$$

$$= E(2\pi rL)$$

But also:

$$\Phi_e = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\rho V_{\text{enclosed}}$$

$$
\begin{aligned}
\Phi_e &= \Phi_e \\
E(2\pi rL) &= \frac{1}{\varepsilon_0}\rho V_{\text{enclosed}} \\
E(2\pi rL) &= \frac{1}{\varepsilon_0}\rho\pi R^2 L \\
E &= \frac{1}{\varepsilon_0}\rho\pi R^2 L\frac{1}{2\pi rL} \\
E &= \frac{\rho\pi R^2 L}{2\varepsilon_0\pi rL} \\
E &= \frac{\rho R^2}{2r\varepsilon_0}
\end{aligned}
$$

So then:

$$\vec{E} = \frac{\rho R^2}{2r\varepsilon_0}\hat{r}$$

Where $\hat{r}$ is the unit vector radataing from the cylinder's axis.

Recap of Gauss's Law

$$\Phi_e = \oint \vec{E}\,\mathrm{d}\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Symmetries

Planar Symmetry

Example: an infinite sheet.

- Translation in any direction along the sheet
- Rotation about an axis perpendicular to the sheet
- Mirror through any perpendicular plane
- Mirror through a plane on the sheet.

Cylindrical Symmetry

Example: Infinite charged rod.

- Translation along the axis
- Rotation about the axis
- Mirror through a plane through the axis
- Mirror through a plane perpendicular to the axis

Spherical Symmetry

Example: Uniformly charged sphere.

- Rotation around any axis that goes through the center of the sphere
- Mirror through any plane that contains the center point

Strategy for applying Gauss's Law

1. Draw a Gaussian surface such that
   1. It has the same symmetry as the electric field
   2. The field is tangent or perpendicular to the surface at every point

Then, integrate!

Uniform charge densities

1. For a line: $\mathrm{d}q = \lambda \, \mathrm{d}L$. λ is in units of charge per length.
2. For a surface: $\mathrm{d}q = \sigma \, \mathrm{d}A$. σ is in units of charge per area.
3. For a volume: $\mathrm{d}q = \rho \, \mathrm{d}V$. ρ is in units of charge per volume.

Example

Consider a uniformly charged sphere with radius R and total charge Q. How much charge is enclosed by a sphere of radius r, with $r < R$.

The answer is $Q\frac{r^3}{R^3}$.

Example

An infinite plane of charge with charge density $+\delta$, the charge per unit area, lies in the xy-plane. Use Gauss's Law to find the electric field above or below the plane.[1]

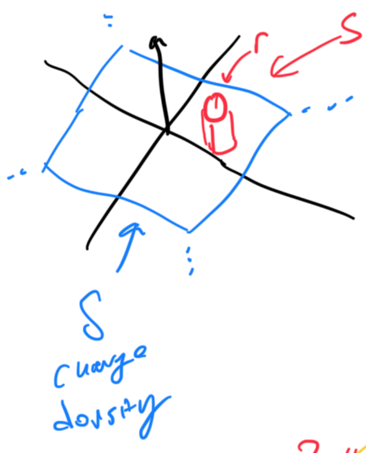
The symmetries show that the charge is perpendicular to the plane.

We chose to use a cylinder because the cylinder only has three sides (convenient!).

Furthermore, the magnitude of the charge on the top and bottom is shown to be the same by mirroring through the plane.

On the side, the charge is parallel, which means there is no flux (dot product is 0). On the top and bottom, E is perpendicular, and therefore $\oint \vec{E} \cdot \mathrm{d}\vec{A} = EA$, since the dot product is 1.

[1] She used σ instead of δ, but I had previously done this problem with δ.



$$\oint_S \vec{E}\cdot d\vec{A} = \oint_{top} \vec{E}\cdot d\vec{A} + \int_{bottom} \vec{E}\cdot d\vec{A} + \oint_{sides} \vec{E}\cdot d\vec{A} \to 0$$

$$= 2\oint_{top} \vec{E}\cdot d\vec{A}$$

$$= 2\cdot \pi r^2 E$$

$$2\pi r^2 E = \frac{E_{enclosed}}{\varepsilon_0} = \frac{\pi r^2 \delta}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\delta}{2\varepsilon_0}$$

Conductors with Gauss's Law

Take a Gaussian surface just inside a conductor's surface. Since there is no charge inside the conductor, the flux is zero, and therefore $Q_{\text{enclosed}} = 0$.

Furthermore, the electric field on a conductor's surface is perpendicular[2].

Then, a Gaussian surface extending through the surface of a conductor has a flux only through the outer face.

The net flux is $\Phi_E = AE_{\text{surface}} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$

Let $Q_{\text{enclosed}} = \eta A$. Therefore, $E_{\text{surface}} = \frac{\eta}{\varepsilon_0}$ and it is perpendicular to the surface.

We derived the field for a plane of charge:

$$E = \frac{\sigma}{2\varepsilon_0}$$

And for a conductor with **surface** charge density η:

$$E = \frac{\eta}{\varepsilon_0}$$

By forming a charged plane out of a very thin conductor, we can see that the formulas agree since, in that situation, $\eta = \frac{\sigma}{2}$.

Holey conductor

Form a hole in a neutral conductor, and put a positive charge of value q there.

[2] see slides for a diagram

Charges remain on the edge only, and the positive charge will attract negative charges to the edge of the hole, of charge $-q$. Then, the charges on the outer surface will be charged q. This is because the conductor was neutral, and therefore the sum of charges in the conductor will remain 0.

This means for any surface that entirely is within a conductor has $Q_{\text{enclosed}} = 0$.

Example

A hollow metal sphere has an inner radius a and an outer radius b. The hollow sphere has the charge $+2Q$. A point charge $+Q$ sits at the center of the hollow sphere. Determine the electric field in the regions $r < a$, $r > b$ and $a < r < b$.

$r < a$:

$$\begin{aligned} \Phi_e = EA &= \frac{Q_{\text{encl}}}{\varepsilon_0} \\ E(4\pi r^2) &= \frac{+Q}{\varepsilon_0} \\ E &= \frac{Q}{4\pi r^2 \varepsilon_0} \end{aligned}$$

$r > b$:

$$\begin{aligned} \Phi_E = EA &= \frac{Q_{\text{encl}}}{\varepsilon_0} \\ E(4\pi r^2) &= \frac{+3Q}{\varepsilon_0} \\ E &= \frac{3Q}{4\pi\varepsilon_0 r^2} \end{aligned}$$

$a < r < b$:

$$E = 0$$

$$\Phi_e = EA = \frac{Q_{\text{encl}}}{\varepsilon_0} = 0$$

$$\begin{aligned} Q_{\text{encl}} &= Q_{\text{inner}} + Q = 0 \\ \rightarrow Q_{\text{inner}} &= -Q \end{aligned}$$

$$\begin{aligned} Q_{\text{inner}} + Q_{\text{outer}} &= 2Q \\ -Q + Q_{\text{outer}} &= 2Q \\ Q_{\text{outer}} &= 3Q \end{aligned}$$

Example

Consider a non-conducting sphere of radius r_1 with a spherical cavity at its center of radius r_0. Assume the total charge Q is distributed uniformly in the "shell" (from $r = r_0 \rightarrow r_1$). Determine the electric field as a function of r for:

1. $r > r_1$

$$EA = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

2. $r_0 < r < r_1$

$$EA = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$$E(4\pi r^2) = Q\frac{V_{\text{charge enclosed}}}{V_{\text{charge}}} = Q\frac{\frac{4}{3}\pi(r^3 - r_0^3)}{\frac{4}{3}\pi(r_1^3 - r_0^3)} = \frac{Q(r^3 - r_0)}{r_1^3 - r_0^3}$$

$$E = \frac{Q(r^3 - r_0)}{4\pi r^2(r_1^3 - r_0^3)}$$

3. $r < r_0$

Important Equations So Far

Coulomb's Law:

$$F = \frac{kq_1q_2}{r^2} = \frac{kq_1q_2}{4\pi\varepsilon_0 r^2}$$

Electric Field:

$$\vec{E} = \frac{\vec{F}}{q}, E = k\frac{Q}{r^2}$$

Superposition of Electric Fields:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

Continous charge distribution:

$$\vec{E} = k\int \frac{1}{r^2}\,\mathrm{d}q$$

It is often useful to write $\mathrm{d}q$ in terms of some small length or area.

Flux for a flat area and uniform $\vec{E}$:

$$\Phi_E = \vec{E}\cdot\vec{A}$$

Flux in general:

$$\Phi_E = \int \vec{E}\cdot\mathrm{d}\vec{A}$$

Gauss's Law:

$$\Phi_E = \oint \vec{E}\cdot\mathrm{d}\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

You often want to construct a Gaussian surface such that the electric field is parallel or perpendicular at all points. This makes calculating the integral via geometry far easier.

Energy Review

- The kinetic energy of a system, K, is the sum of all kinetic energies $K_i = \frac{1}{2}m_i v_i^2$, for all particles in the system.
- The potential energy of the system, U, is the interaction energy of the system
- The change in potential energy, ΔU, is the negation of the amount of work done by interaction forces.
- If all forces involved are conservative, then the total energy of the system $K + U$ remains constant.

Analogy

Any conservative force can be given a potential energy. For gravity, this will be:

$$\Delta U = -W$$
$$W = mgy_i - mgy_f$$

Therefore:

$$U = U_0 + mgy$$

If an object falls, this means that $\Delta U < 0$, and gravity did work, so $W > 0$.

Potential Energy

A positive charge q inside a capacitor speeds up as it falls toward the negative plate.

This force, $F = qE$, is constant because the electric field is also constant. Therefore, the work is $W = Fd = qE(s_i - s_f)$.

Define s as the perpendicular distance to the negative plate. Then, $U_{\text{elec}} = U_0 + qEs$.

Aside: Charged Plates

If you have two oppositely charged (infinite) plates with charge densities σ and $-\sigma$, the electric field between the plates is $E = \frac{\sigma}{\varepsilon_0}$, and 0 outside the plates.

Work

$$W = \int_{\text{pos}_i}^{\text{pos}_f} \vec{F} \cdot \mathrm{d}\vec{\ell}$$

For the electric field, this becomes

$$W = q \int_{\text{pos}_i}^{\text{pos}_f} \vec{E} \cdot \mathrm{d}\vec{\ell}$$

As always, $\Delta U = -W$.

Example

Consider two like charges, q_1 and q_2, with q_1 fixed.

Then, the work done by the electric field becomes:

$$
\begin{aligned}
W_{\text{elec}} &= \int_{x_1}^{x_2} F_{1\to 2}\,\mathrm{d}x \\
&= \int_{x_1}^{x_2} \frac{kq_1q_2}{x^2}\,\mathrm{d}x \\
&= -kq_1\frac{q_2}{x}\bigg|_{x_1}^{x_r}
\end{aligned}
$$

This results in:

$$U = \frac{kq_1q_2}{r}$$

This uses the zero point of infinitely far apart for similar reasons to why gravity does so.

Problem

An interaction between two elementary particles causes an electron and a positron to be shot out in opposite directions with equal speeds. What speed must each have when they are 100 fm apart to escape each other?

We should use conservation of energy.

The final potential energy is 0 because we defined the zero point of an electric field's effect to be at infinity, which is when they have escaped.

$$
\begin{gathered}
\frac{1}{2}m_ev_i^2 + \frac{1}{2}m_ev_i^2 + \frac{kq_1q_2}{r} = \frac{1}{2}m_ev_f^2 + \frac{1}{2}m_ev_f^2 + \frac{kq_1q_2}{\infty} \\
\frac{1}{2}m_ev_i^2 + \frac{1}{2}m_ev_i^2 + \frac{kq_1q_2}{r} = 0 \\
m_ev_i^2 + \frac{kq_1q_2}{r} = 0 \\
m_ev_i^2 = -\frac{k(1e^-)(-1e^-)}{r} \\
v_i^2 = -\frac{k(1e^-)(-1e^-)}{m_er} \\
v_i = \sqrt{-\frac{k(1e^-)(-1e^-)}{m_er}} \\
v_i = \sqrt{\frac{k(1e^-)(1e^-)}{m_er}}
\end{gathered}
$$

Therefore, $v_i = 5.03 \times 10^7\frac{m}{s}$.

Multiple Charges

Consider more point charges. Then, the potential energy is the sum of all the potential energies due to all pairs of charges:

$$U = \sum_{i<j} \frac{kq_iq_j}{r_{i,j}}$$

Where $r_{i,j}$ is the distance between q_i and q_j.

Problem

Three point charges, which are initially infinitely far apart, are placed at the corners of an equilateral triangle with sides d. Two of the point charges are identical and have charge q. If zero net work is required to place the charges in the corners of the triangle, what must the value of the third charge be?

Initally, $U = 0$

At the end, $r_{i,j} = d$.

$$\begin{aligned} & \sum_{i<j} \frac{kq_i q_j}{r_{i,j}} \\ = & \sum_{i<j} \frac{kq_i q_j}{d} \\ = & \frac{k}{d}(q_1 q_2 + q_1 q_3 + q_2 q_3) \\ = & \frac{k}{d}(q^2 + qq_3 + qq_3) \\ = & \frac{qk}{d}(q + 2q_3) \end{aligned}$$

$$\begin{aligned} \frac{qk}{d}(q + 2q_3) &= 0 \\ q + 2q_3 &= 0 \\ 2q_3 &= -q \\ q_3 &= -\frac{q}{2} \end{aligned}$$

Defining the Electric Potential

V, electric potential, is defined as:

$$V = \frac{U_{q\,+\,\text{sources}}}{q}$$

The SI unit for electric potential is the volt $V = J/C$.

Therefore, the potential due to a point charge is:

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

r is the distance from the point charge to the place of measurement.

Like the electric field, the electric potential is independent of the test charge.

For a collection of point charges, simply sum the contribution from each point charge:

$$V = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i}$$

.

If you move in the direction of the electric field, the electric potential decreases, and opposite, it decreases.

The general relation between the field and potential can be constructed from work, resulting in:

$$\Delta V = -\int_a^b \vec{E} \cdot \mathrm{d}\vec{s}$$

For the specific case of a uniform field, this simplifies to:

$$V_{ba} = -Ed$$

Equipotential surfaces are perpendicular to the electric field at all points.

As a charge moves through a changing electric potential, energy is conserved:

$$K_f + qV_f = K_i + qV_i$$

Therefore, if $\Delta V > 0$, positive charges will slow down and if $\Delta V < 0$, positive charges will speed up.

This then means that the preferred position for a positive charge is the location that has the most negative potential.

The reverse is true for negative charges.

Summary

- Electric force is conservative
  - $\Delta K + \Delta U = 0$
  - $\Delta U = -W$
- Work Defintion
  - $\Delta U = -W = -\int_{\vec{i}}^{\vec{f}} \vec{F} \cdot \mathrm{d}\vec{s}$
- Electric Potential
  - $\Delta V = \frac{\Delta U}{q} = -\int_{\vec{i}}^{\vec{f}} \vec{E} \cdot \mathrm{d}\vec{s}$
- For point charges
  - $U = \frac{kq_1q_2}{r}$
  - $U = \sum_{i<j} \frac{kq_iq_j}{r_{i\to j}}$
  - $V = \frac{kq}{r}$
  - $V = \sum_i \frac{kq_i}{r_i}$
- For a charge distribution:
  - $V = \int \frac{k}{r}\,\mathrm{d}q$
- Find E from V:
  - $\vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$

A separation of charge creates an electric potential difference, because it creates an electric field.

Example

A very long conducting cylinder of length ℓ of radius R_0, $R_0 \ll \ell$, carries a uniform surface charge density η. The cylinder is at an electric potential V_0. Determine the potential, at points far from the end, at a distance r from the center of the cylinder for:

1. $r > R_0$
2. $r < R_0$

Find $V(r)$.

First we will find $E(r)$, using gauss's law, assuming $r > R_0$:

Assume a cylinder wrapping the cylinder of length L and of radius r.

$$\Phi_e = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E \times 2\pi r L = \frac{Q_{\text{enclosed}}}{\varepsilon_0} = \frac{(2\pi R_0 L)\eta}{\varepsilon_0}$$

Therefore, overall:

$$\vec{E} = \frac{\eta R_0}{\varepsilon_0 r}\hat{r}$$

For $r < R_0$:

$$\vec{E} = 0$$

.

Find V from E for $r > R_0$.

$$\Delta V = V_f - V_i = -\int_{\vec{i}}^{\vec{f}} \vec{E} \cdot \mathrm{d}\vec{s}$$

$$V(r) - V_0 = -\int_{R_0}^{r} \frac{\eta R_0}{\varepsilon r'}\,\mathrm{d}r'$$

$$V(r) - V_0 = -\frac{\eta R_0}{t_0}\ln(r')|_{R_0}^{r}$$

$$V(r) = V_0 - \frac{\eta R_0}{t_0}\ln\left(\frac{r}{R_0}\right)$$

For $r < R_0$

$$\Delta V = V_f - V_i = -\int_{\vec{i}}^{\vec{f}} \vec{E} \cdot \mathrm{d}\vec{s}$$

$$V(r) - V_0 = 0 \to V(r) = V_0$$

Furthermore, as a general point, the potential is constant in regions where $\vec{E} = 0$.

Example

A finite rod of length $2L$ has a total charge q, distributed uniformly along its length. Consider the rod as on the x-axis and centered at the origin. Thus, one endpoint is located at $(-L, 0)$, and the other at $(L, 0)$. Define the electric potential to be zero at an infinite distance away from the rod. Point A is located at $(0, y)$. What is V_A, the electric potential at point A?

$$V = \frac{kq}{2L}\ln\left(\frac{\sqrt{L^2 + y^2} + L}{\sqrt{L^2 + y^2} - L}\right)$$

Find E from V

$$\vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

Alternatively,

$$\vec{E} = -\nabla V(x, y, z)$$

Example

Determine the electric field vector in a region of space if the potential varies as follows:

$$V = ay^2 + bxy - cxyz$$

Solution

$$\vec{E}_x = -\frac{\partial V}{\partial x} = -(by - cyz) = cyz - by$$

$$\vec{E}_y = -\frac{\partial V}{\partial y} = -(2ay + bx - cxz) = cxz - 2ay - bx$$

$$\vec{E}_z = -\frac{\partial V}{\partial z} = -(-cxy) = cxy$$

Example

Continuing from the previous problem with the finite rod of length $2L$:

$$V = \frac{kq}{2L} \ln\left(\frac{\sqrt{L^2 + y^2} + L}{\sqrt{L^2 + y^2} - L}\right)$$

We then can know that $E_y = \frac{kq}{y\sqrt{y^2+L^2}}$ by taking the derivative and using the symmetry of the problem to get the direction. This gives the same answer as it done the traditional way.

Equipotential Surfaces

They are always the surface perpendicular to electric field vectors.

In conductors:

- When all charges are at rest:
  - The surface of a conductor is an equipotential surface.
  - The electric field outside a conductor is, therefore, perpendicular to the surface
  - the entire volume of the conductor has the same potential as the surface.

Capacitors

Any two conductors separated by an insulator (or a vacuum) form a capacitor.

When the capacitor is charged, the conductors have equal magnitude but opposite signs, so the net charge is zero on the capacitor.

A common way to charge a capacitor is to connect the conductors to opposite terminals of a battery.

If we change the magnitude of the charge on each conductor, the potential difference between conductors changes; however, the ratio of charge to potential difference does not change.

This ratio is called the capacitance of the capacitor:

$$C = \frac{Q}{V_{ab}}$$

This is units of the farad: $1F = 1\frac{C}{V}$.

Parallel-plate capacitor

A parallel-plate capacitor consists of two parallel conducting plates separated by a distance that is small compared to their dimensions.

The field between the plates of a parallel-plate conductor is considered uniform, and the charges on the plates are uniformly distributed. When the region between the plates is empty, capacitance can be calculated from the field:

- $V_{ab} = Ed$
- $E = \frac{\eta}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$
- $V_{ab} = \frac{Qd}{\varepsilon_0}$
- $C = \frac{\varepsilon_0 A}{d}$
- The capacitance depends only on the geometry of the capacitor.

Cylinder Capacitor

A cylindrical capacitor consists of a cylinder (or wire) of radius R_b surrounded by a coaxial cylindrical shell of inner radius R_a. Both cylinders have length ℓ, which we assume is much greater than the separation of the cylinders, so we can neglect the end effects. The capacitor is charged (by connecting it to a battery) so that one cylinder has a charge $+Q$ (say, the inner one) and the other one a charge $-Q$. Determine a formula for the capacitance.

$$C = \frac{Q}{V}$$

$$V_{ba} = V_b - V_a = -\int_a^b \vec{E} \cdot \mathrm{d}\vec{s}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{Q}{2\pi\varepsilon_0 r\ell}$$

$$V_{ab} = V_a - V_b = -\frac{Q}{2\pi\varepsilon_0 \ell} \ln\left(\frac{R_a}{R_b}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 \ell}{\ln\left(\frac{R_a}{R_b}\right)}$$

One common way to charge a capacitor is to connect the two conductors to opposite terminals of a battery. (batteries create a ΔV).

If we change the magnitude of the charge on each conductor, the potential difference changes, but not the ratio of charge to potential difference. That ratio, $\frac{Q}{\Delta V}$ is the capacitance.

Two parallel lines indicate a capacitor: ⊣⊢

Capacitors in parallel have the same potential, V.

Then, the charge on each depends on the capacitance:

$$V_1 = \frac{Q_1}{C_1} \quad V_2 = \frac{Q_2}{C_2} \Longrightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

This can be simplified into a single capacitor:

$$C_{\equiv} = \frac{Q_{\equiv}}{V_{ab}} = C_1 + C_2$$
$$Q_{\equiv} = Q_1 + Q_2$$

For capacitors in series, their potential differences add: $V_{ac} + V_{cb} = V_{ab}$, and they have the same charge Q.

$$V_1 = \frac{Q_1}{C_1}$$
$$V_2 = \frac{Q_2}{C_2}$$
$$V_{ab} = V_1 + V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

This can be simplified into a single capacitor:

$$Q_{\equiv} = Q$$
$$C_{\equiv} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Energy stored in a capacitor

The work needed to add a small amount of charge when the potential difference between capacitor plates is V is $\mathrm{d}W = V\,\mathrm{d}q$.

Then, we know $V = \frac{q}{C}$. Therefore:

$$W = \int \mathrm{d}W = \int_0^Q V\,\mathrm{d}q = \frac{1}{C}\int_0^Q q\,\mathrm{d}q = \frac{Q^2}{2C}$$

This then means that:

$$U = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{QV}{2}$$

For a parallel plate capacitor, we can plug some things in:

$$V = Ed, \quad C = \frac{\varepsilon_0 A}{d} \Longrightarrow U = \frac{1}{2}\varepsilon_0 E^2 Ad$$

Then, we can calculate, u, the energy per unit volume:

$$u = \frac{U}{d \times A} = \frac{1}{2}\varepsilon_0 E^2$$

This is also the energy density for any electric field (not just for parallel plate capacitors).

Dielectrics

- Most capacitors have a nonconducting material (called a dielectric) between the conducting plates
- A common capacitor design uses long strips of metal foil for the plates, which are separated by strips of plastic sheet.

When an insulating material is inserted between the plates of a capacitor whose original capacitance is C_0, the new capacitance is greater by a factor K, where K is the dielectric constant of the material:

$$C = C_0 K = K\varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d}$$

People use dielectrics because they increase energy densities and allow for the storage of higher voltages.

- When a dielectric is inserted between the plates of a capacitor, the electric field decreases
  - $E = E_0 / K$
- This is due to the polarization of the charge within the dielectric, which results in induced surface charges.

Unfortunately, we live in the real world and things break down. In the case of dielectrics, they can become a conductor. The dielectric strength is the maximum electric field the material can withstand. This measurement is in units of $\frac{V}{m}$. For example, for pyrex, its $E_m = 1 \times 10^7 \ \mathrm{V} / \mathrm{m}$.

If you charge a capacitor, disconnect it, and then insert a dielectric, this will decrease the potential since the capacitance increases and the charge remains constant.

If it had remained connected, this would have increased the charge since the voltage is constant and the capacitance increased.

Problem

A capacitor with capacitance C_1 is charged by a battery with voltage V_0. It is disconnected from the battery and then connected to an uncharged capacitor with capacitance C_2.

1. Determine the total stored energy before the two capacitors are connected.

$$U_i = \frac{1}{2} C_1 V_0^2$$

$$C_{\equiv} = C_1 + C_2$$
$$Q_{\equiv} = Q_{1i} = C_1 V_0$$

   The initial charge Q_{i1} is the total final charge for C_1 and C_2.

   Final voltages:

$$V_{1f} = \frac{Q_{1f}}{C_1} = V_{2f} = \frac{Q_{2f}}{C_2} = V_{\equiv} = \frac{Q_{\equiv}}{C_{\equiv}} = \frac{C_1 V_0}{C_1 + C_2}$$

2. Determine the total stored energy after they are connected.

$$U_f = \frac{1}{2} C_1 \left(\frac{C_1 V_0}{C_1 + C_2} \right)^2 + \frac{1}{2} C_2 \left(\frac{C_1 V_0}{C_1 + C_2} \right)^2 = \frac{C_1^2 V_0^2}{2(C_1 + C_2)}$$

3. What is the change in energy?

$$\Delta U = U_f - U_i = -\frac{C_1 C_2 V_0^2}{2(C_1 + C_2)}$$

4. What is the charge on each of the capacitors after they are connected?

Review

Capacitors in parallel:

$$V_{\equiv} = V_1 = V_2 = V_3 = \cdots$$
$$Q_{\equiv} = Q_1 + Q_2 + Q_3 + \cdots$$
$$C_{\equiv} = C_1 + C_2 + C_3 + \cdots$$

Capacitors in series:

$$V_{\equiv} = V_1 + V_2 + V_3 + \cdots$$
$$Q_{\equiv} = Q_1 = Q_2 = Q_3 = \cdots$$
$$\frac{1}{C_{\equiv}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

This differs from the last class because it shows the generalization.

For dielectrics:

$$C = C_0 K = K\varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d}$$

Energy Density:

$$u = \frac{U}{d \times A} = \frac{1}{2}\varepsilon_0 E^2$$

Conduction

- Connecting a wire to a battery causes a nonuniform surface charge distribution
- Surface charges induce an electric field inside the wire
- The electric field pushes the sea of electrons through the metal.
- Electrons are what actually move and carry the charge, but traditionally you treat current as the motion of positive charges.

Kirchhoff's Junction law

The current is the same everywhere in a circuit without junctions, and the sum of currents entering a junction equals the sum leaving.

Resistance

Collisions of electrons with atoms cause a conductor to resist the motion of charges.

Resistivity is an electrical property of a material like copper.

Resistance is a property of a *specific* wire or circuit based on what is made of *and* its size and shape.

Ohm's Law

The current flowing through a wire or circuit element depends on the potential difference and the resistance.

$$I = \frac{\Delta V}{R}$$

Electromotive Force

Electromotive Force (not a force) is similar to a pump for water.

This is denoted as EMF and is the influence that makes current flow from lower to higher potential. A circuit device that provides emf is called a source of emf.

The symbol $\mathcal{E}$ is for emf.

ν

Batteries

Batteries are a source of emf and transform chemical energy into electrical energy. Electricity can be created if dissimilar metals are connected by a conductive solution called an electrolyte, creating a simple electric cell.

As the cathode (-) gets dissolved by the electrolyte, each atom leaves 2 (in this case) electrons on the electrode and positive zinc ions enter the electrolyte.

The electrolyte then becomes positively charged and can pull electrons off the anode. As electrons are pulled off the anode, it becomes positively charged.

Then, the equal and opposite charges on the cathode and anode create a potential difference between the terminal ends.

Current

A current is any motion of charge from one region to another.

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

The units of current is the amp, $A = C/s$.

Conventional current is treated as a flow of positive charges.

In a metallic conductor, the moving charges are electrons, but the current still points in the direction in which positive charges would flow.

Current density:

$$\vec{J} = nq\vec{v}_d$$

n is the concentration of moving charged particles. q is the charge per particle. $\vec{v}_d$ is the drift velocity.

$\left[\vec{J}\right] = C/(m^2 \cdot s)$

The resistivity of a material is:

$$\rho = \frac{E}{J}$$

The conductivity is the reciprocal of the resistivity:

$$\sigma = \frac{1}{\rho}$$

Material	$\rho \quad (\Omega \cdot m)$
Copper	1.72×10^{-8}
Gold	2.44×10^{-8}
Glass	10^{10}– 10^{14}

The resistivity varies with temperature and usually increases with higher temperatures.

However, for semiconductors, it decreases with higher temperatures.

For superconductors, if the temperature gets low enough, the resistivity becomes zero.

The resistance of a conductor is:

$$R = \frac{\rho L}{A}$$

A is the cross-sectional area.

Then, Ohm's law can give you the potential with $\Delta V = IR$.

In many conductors, the resistance is independent of the voltage and related by Ohm's law, $V = IR$.

Resistance is in ohms, $1\Omega = 1V/A$

If a material does not follow Ohm's law, it is called nonohmic.

Note the Δ. This means that after a resistor, the potential decreases.

Summary

- Batteries maintain a constant potential difference (the current may vary)
- Resistance is a property of a specific device
- Current is not a vector, but has direction
- Current and charge do not get used.

Example

A particular wire has a length $L = 1.5 \text{ meters}$ and a circular cross-sectional area of $r = 2.0 \text{ mm}$. The resistance of this wire is 25Ω. What is the resistivity?

$$R = \frac{\rho L}{A} \to \rho = \frac{RA}{L} = \frac{R(\pi r^2)}{L} = 0.00020944\ m\Omega \Rightarrow \sigma = 4774.65\ 1/(m\Omega)$$

$$J = \sigma E = \frac{E}{\rho} = \frac{EL}{\pi R r^2}$$

Calculate the current in the wire if the field strength is 7.0 V/m.

$J = \frac{I}{A} \Rightarrow I = JA \Rightarrow \frac{EL}{RA} A = \frac{EL}{R} = 420 \text{ mA}$

If you have a device that has a potential difference across it, $V_{ab} = V_a - V_b$, and then $P = V_{ab} I$. This is where current passes from a towards b.

In general, $P = IV$, and for an ohmic resistor, this becomes $P = I^2 R$.

Example

Consider two incandescent bulbs. Bulb 1 has a resistance that is twice as large as bulb 2, but they are otherwise similar. The voltage drop is the same for both. Which is brighter?

Assume brightness $\propto$ power.

$R_1 = 2R$, $R_2 = R$.

$P_1 = I_1 V = \frac{V^2}{2R}$, $P_2 = I_2 V = \frac{V^2}{R}$

Since $P_2 > P_1$, bulb 2 is brighter.

Alternatively, assume the current flowing through them is the same.

$R_1 = 2R$, $R_2 = R$
$P = I^2 R$ by $V = IR$

Therefore, bulb 1 will be brighter if the current remains the same.

Example

A power station delivers 750 kW of power at 12,000 V to a factory through wires with a total resistance of 3.0 Ω. How much less power is wasted if the electricity is delivered at 50,000 V instead of 12,000 V?

$$P_{\text{diss}} = I^2 R = \left(\frac{P}{V}\right)^2 R$$

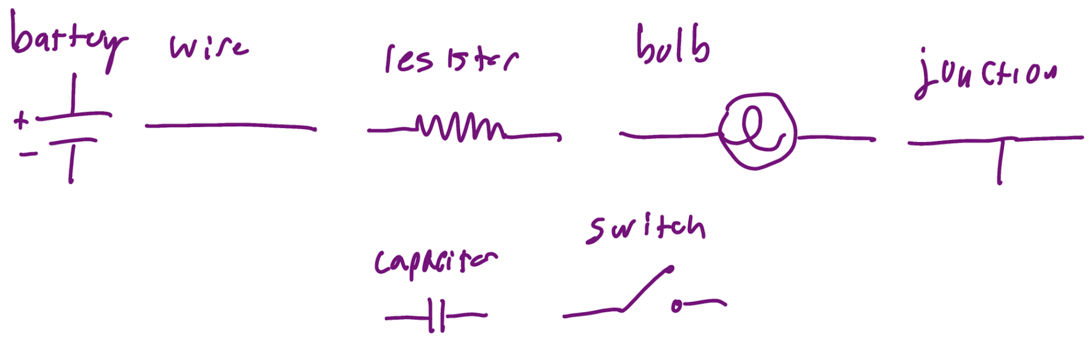
$$\Delta P_{\text{diss}} = \left(\frac{P}{V_1}\right)^2 R - \left(\frac{P}{V_2}\right)^2 R = RP^2\left(\frac{1}{V_1^2} - \frac{1}{V_2^2}\right) = 11.0438 \text{ kW}$$

Circuit Diagrams

A logical picture of what is connected to what. The precise mechanism of the connection is not specified.

- the longer end of the battery symbol indicates the positive terminal and the emf of the battery might be written next to it
- wire should be assumed to have no resistance

Circuit elements:



Resistor

Potential decreases across a resistor if you travel in the direction of the current. In other words, if point A is further down the line to point B, $V_A > V_B$.

Furthermore, $V_A - V_B = IR$.

In Series

In series, resistors add their resistances.

By Kirchhoff's junction law, the current is the same across all resistors: $I = I_1 = I_2 = I_3 = \cdots$.

We furthermore know that the voltages add: $V_{\equiv} = V_1 + V_2 + V_3 + \cdots$

$$R_{\equiv} = R_1 + R_2 + R_3 + \cdots$$

In parallel

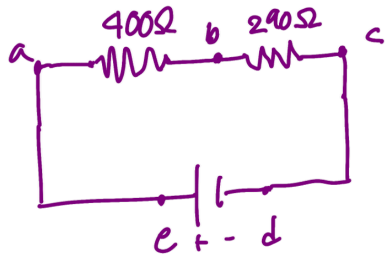
If the resistors are in parallel, the potential difference is constant: $V_{\equiv} = V_1 = V_2 = V_3 = \cdots$. Furthermore, the currents add: $I_{\equiv} = I_1 + I_2 + I_3 + \cdots$.

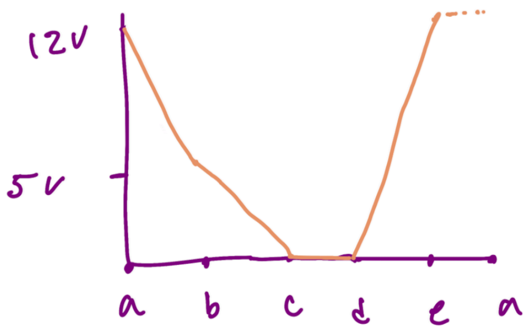
Together this becomes:

$$\frac{1}{R_{\equiv}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

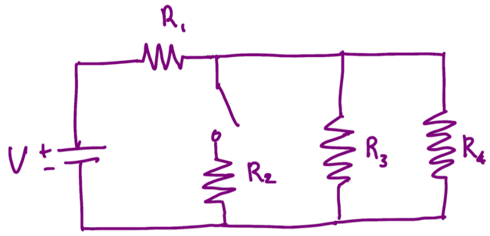
Something to think about is that the electrons will "choose" the path of least resistance, and resistors resist because electrons bump into stuff, so parallel resistors will have a lower resistance than any of the individual resistors.

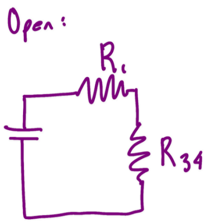
Example

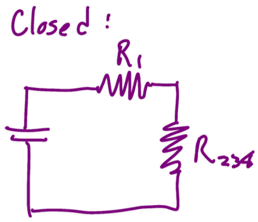




Example







What happens to the voltage across each resistor when the switch is closed?

We know that adding a resistor will decrease the resistance, so $R_{234} < R_{34}$. This will also therefore decrease the overall resistance, which increases the overall current.

Let I be the overall current.

$$V_1 = R_1 I$$
$$V_{2?34} = V - V_1 = V - R_1 I$$

V is constant, but since I increased, V_1 increased, therefore making $V_{2?34}$ decrease, which furthermore implies V_3 and V_4 decreased. Since V_2 was previously zero, it must have increased.

What happens to the current through each resistor when the switch is closed?

By above, we know that the voltage through V_1 and V_2 increased, and the voltage through V_3 and V_3 decreased. Since $V = IR \to R = \frac{V}{I}$, I_1, I_2 decreased and I_3, I_4 increased.

What happens to the power output of the battery when the switch is closed?

Since $P = IV$, and the voltage stays constant but the current drops, the power also drops after the switch is closed.

Let $R_1 = R_2 = R_3 = R_3 = 135\Omega$ and $V = 22.0 \text{ V}$. Determine the current through each resistor before and after closing the resistor.

Open:

$$R_{\equiv} = R_1 + \left(\frac{1}{R_3} + \frac{1}{R_3}\right)^{-1}$$

$$I_{23} = I_1 = I = \frac{V}{R_{\equiv}}$$

$$V_1 = IR_1$$

$$R_{23} = \left(\frac{1}{R_3} + \frac{1}{R_3}\right)^{-1}$$

$$V_{23} = IR_{23}$$

$$V_{23} = V_2 = I_2 R_2 \to I_3 = \frac{V_{23}}{R_2}$$

Alternative:

$$I_4 = I - I_3 = I - I_4 \Rightarrow I_4 = \frac{I}{2} = I_3$$

Closed:

$$R_{\equiv} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_3}\right)^{-1}$$

$$I_{234} = I_1 = I = \frac{V}{R_{\equiv}}$$

$$I_2 = I_3 = I_4 = \frac{I_{234}}{3}$$

Batteries

In reality, batteries have some resistance through them.

This then results in $V_{ab} = \mathcal{E} - Ir$.

$\mathcal{E}$ is the battery's emf, r is the internal resistance, and I is the current through the battery.

Kirchhoff's rules

- For any junction, $\sum I = 0$.
- For any closed loop, $\sum V = 0$.
  - Conventionally, $+\mathcal{E}$ of potential difference is created going from the negative to the positive terminal.
  - It doesn't actually matter! But stay consistent.

Resistors in series

If you have resistors in series:

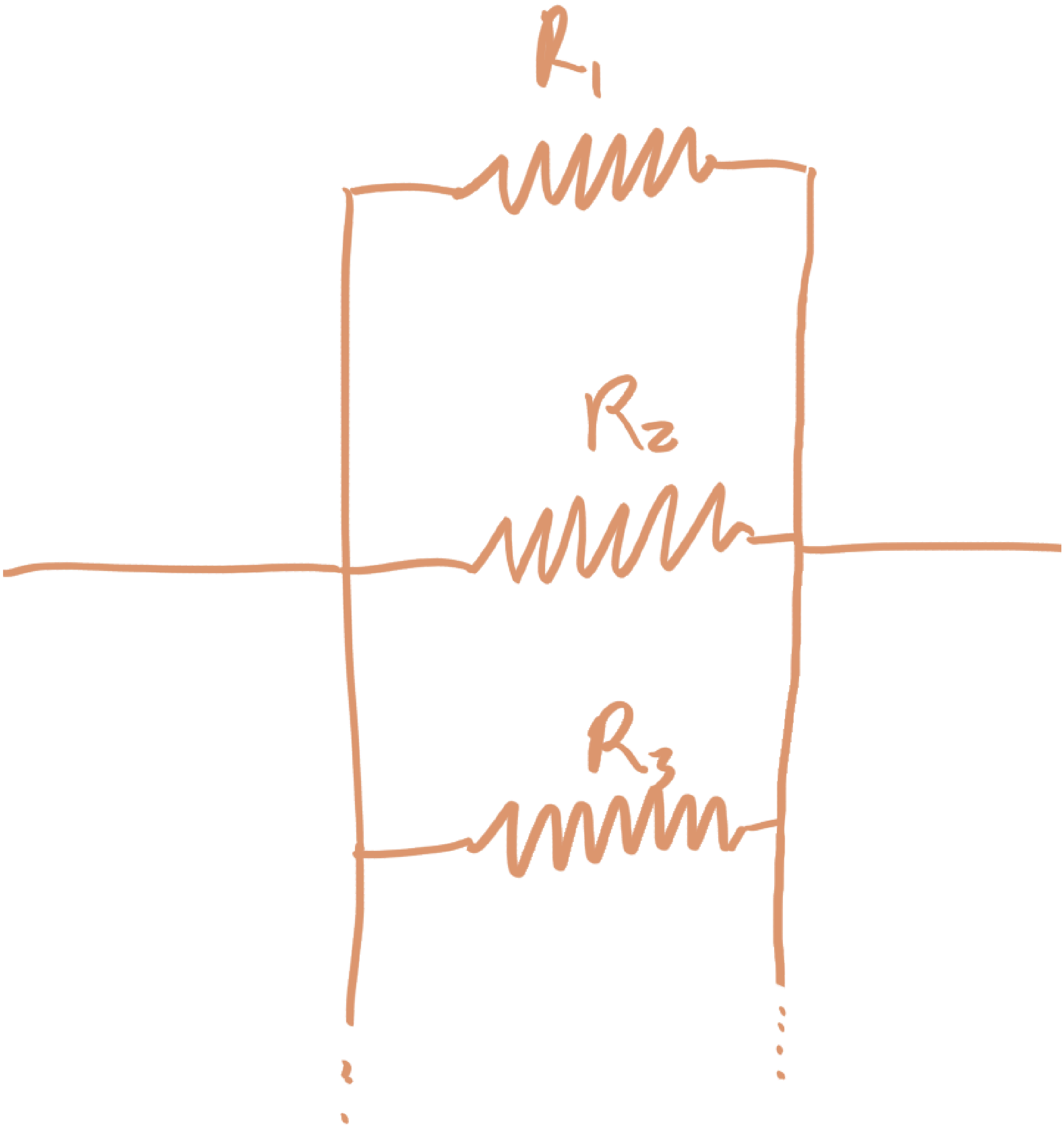
The resistances are added, the voltage difference are added, and the current remains constant.

$$R_{\equiv} = R_1 + R_2 + R_3 + \cdots$$

For resistors in parallel:

The currents add the voltage difference is constant, and the resistance is averaged by the harmonic mean.

$$\frac{1}{R_{\equiv}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$



Current Calculations

Use Kirchhoff's junction rule. The sum of all currents at any junction is zero.

$$\sum I = 0$$

Voltage Calculations

A loop is any closed conducting path. Kirchhoff's loop rule, which is valid for any closed loop, is:

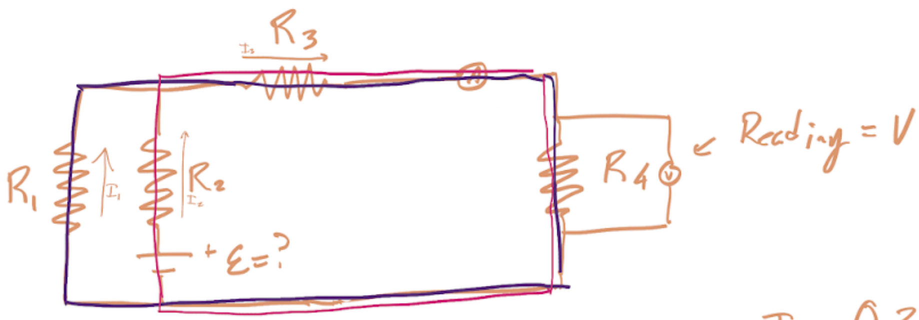
$$\sum V = 0$$

where each V is a potential difference across some segment. The loop rule is a statement that the electrostatic force is conservative.

Example

In the circuit shown in the figure, the batteries have negligible internal resistance, and the meters are both idealized. With the switch open, the voltmeter reads $15.0\ \mathrm{V}$.

1. Find the emf of the battery.
2. What will the ammeter read when the switch is closed?



$R_1 = 75.0\,\Omega$

$R_2 = 20.0\,\Omega$

$R_3 = 30.0\,\Omega$

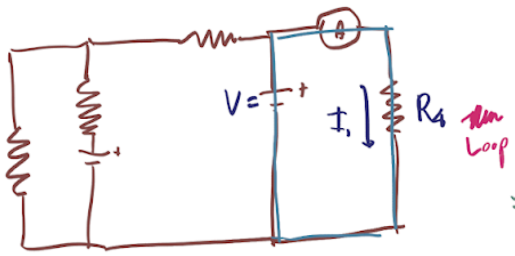
$R_4 = 50.0\,\Omega$

$I_3 \cdot R_4 = V \Rightarrow I_3 = 0.3A$

$\Rightarrow -I_1R_1 - I_3R_3 - I_3R_4 = 0$ Loop

$\Rightarrow I_1 = \frac{-80\,\Omega \cdot 0.3A}{75\,\Omega} = -0.32A$

$I_1 + I_2 - I_3 = 0 \Rightarrow I_2 = I_3 - I_1 = 0.62A$



$\varepsilon - I_2R_2 - I_3R_3 - I_3R_4 = 0$

$\Rightarrow \varepsilon = I_2R_2 - I_3(R_3 + R_4)$

$\varepsilon = 36.4V$

$V - I_1R_4 = 0$

$\Rightarrow V = I_1R_4 \Rightarrow I_1 = \frac{V}{R_4} = \frac{25V}{50\,\Omega} = 0.5A$

Example

We idealize the battery to have a constant emf and zero internal resistance, and we ignore the resistance of connecting conductors.

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

$$\mathcal{E} - \frac{\mathrm{d}q}{\mathrm{d}t}R - \frac{q}{C} = 0$$

$$\Rightarrow q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right)$$

Example

Discharging a circuit.

$$i = -\frac{\mathrm{d}q}{\mathrm{d}t}$$

Here, we have $q(0) = Q_0$.

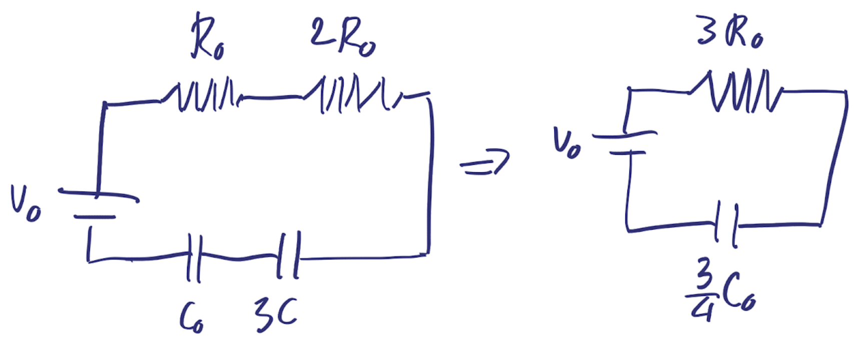
$$ir - \frac{q}{C} = 0$$

$$-\frac{\mathrm{d}q}{\mathrm{d}t}r - \frac{q}{C} = 0$$

$$\Rightarrow q(t) = Q_0 e^{-t/(CR)}$$

Battery

A battery with voltage V_0 is connected at time $t = 0$ to two resistors with resistances R_0 and $2R_0$ in series with 2 capacitors with capacitances C_0 and $3C_0$. Find an equation for the current in the circuit as a function of time in terms of the parameters given.



$$\Rightarrow q(t) = \frac{3}{4}C_0\left(1 - e^{-4t/(9R_0C_0)}\right)$$

Review Day

Exam Format

Practice Exam exists on Canvas. You probably should do it!

- 4 conceptual problems
- 2 shorter problems
- 2 longer problems

You are allowed one 8.5″x11″ formula sheet. There will not be one provided. If your sheet is double-sided, you will not be allowed to use it.

Most problems will have answers in the form of variables, but at least one question will have numbers, requiring a ***calculator***.

Showing your work is good for partial credit and may be required.

Subparts are there to measure your knowledge of getting to the answer.

Many problems will be very similar. At least one will be "*very similar* to one done in a lecture or on the homework. Probably more than one."

There will probably be a problem involving an integral to find either E or V.

On the slides, there's a diagram of the calculus knowledge you need.

Constants are given on the exam.

Concepts

Coulomb's Law

$$F = k\frac{|q_1 q_2|}{r^2}$$

$$k = 8.99 \times 10^9 \text{ N m}^2/\text{C }^2$$

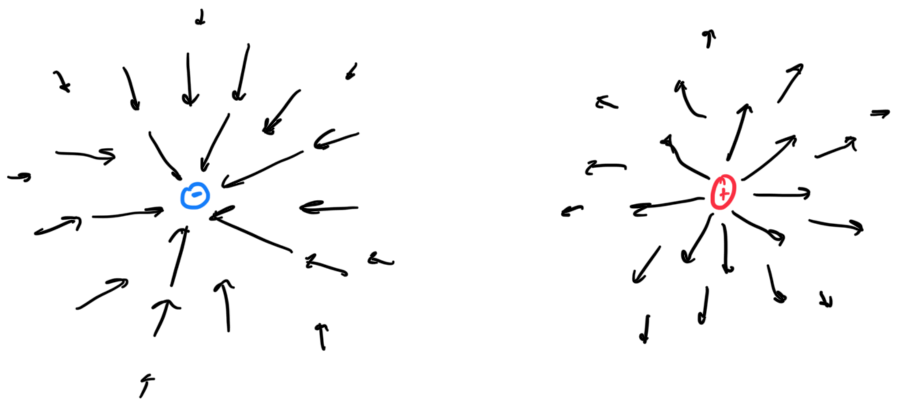
The direction of the force is dependent on the charges. Like charges repel, opposites attract.

Electric Field

From a single point charge:

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

$\hat{r}$ is radially out from the source charge. This means that for positive charges, the electric field is radially outward, and for negative charges, it is radially inward.



Furthermore, by Coulomb's law, $\vec{F} = \vec{E}q$.

For multiple charges, you can sum the electric fields coming from those charges.

For a continuous distribution of charge, act as if it is infinitely many point-like charges:

$$\vec{E} = \int \frac{k}{r^2}\hat{r}\,\mathrm{d}q$$

Usually, this can be made easier through symmetries to find the direction or eliminate annoying parts of the problem.

Types of Problems

Calculate forces and electric fields for multiple point charges.

Calculate the motion of a charged particle in a uniform electric field.

Calculate the electric field at a point from a continuous charge distribution.

Gauss's Law

Flux is

$$\Phi_E = \int \vec{E} \cdot \mathrm{d}\vec{A}$$

This can usually be made simplier through special cases:

- E constant and $\perp$ to surface: $\Phi_E = EA$
- E constant and $\parallel$ to surface: $\Phi_E = 0$

For a closed surface:

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Strategies

1. Determine the symmetry of the electric field.
2. Draw a gaussian surface with the same symmetry as the electric field such that it is either perpendicular or parallel to the electric field at every point.

Inside a conductor, $\vec{E} = \vec{0}$, which implies that the voltage throughout a conductor is constant.

Types of problems

- The electric field for spheres, cylinders, and sheets of charge.
- the electric field for charge distributed uniformly through a volume with an above symmetry (e.g. slab of charge)
- Calculate how charge gets distributed on the inner and outer surfaces of conductors ($\vec{E} = \vec{0}$ and all of the charges are on the outside of the conductor.)
  - $Q_{\text{inner}} + Q_{\text{outer}} = Q_{\text{total}}$
  - Since for closed surfaces, $Q_{\text{enclosed}}/\varepsilon_0 = \oint \vec{E} \cdot \mathrm{d}\vec{A}$, and $\vec{E} = \vec{0}$ in a conductor, so $Q_{\text{enclosed}} = 0$.

Electric Potential

$$V_{\text{ba}} = \frac{U_b - U_a}{q} = -W_{\text{ab}}$$

W_{ab} is the work done by $\vec{E}$ to move a charge from a to b.

$$V_{\text{ba}} = V_b - V_a = -\int_a^b \vec{E} \cdot \mathrm{d}\vec{\ell}$$

Point charges, setting $V = 0$ to $r = \infty$:

$$V = \frac{kQ}{r}, \quad U = \frac{kQq}{r}$$

Charge distrubtion:

$$V = \int \mathrm{d}V = \int \frac{k}{r} \, \mathrm{d}q$$

$$\vec{E} = -\nabla V(x, y, z)$$

Note that there are two types of integrals for finding V. You should use the first one if you know the electric field already. If you have a continuous charge distribution, use the $\int \frac{k}{r} \, \mathrm{d}q$ one. Note that the first one does not assume $V = 0$ at $r = \infty$, but the second one does.

Types of Problems

- Calculate the motion of charged particles via conservation of energy.
- Calculate V from E.
- Calculate V from a continuous distrubtion of charge.
- Calculate E from V via partial derivatives.

Capacitor

Capacitance:

$$C = \frac{Q}{V}, \quad Q = CV, \quad V = \frac{Q}{C}$$

For a parallel plate capacitor:

$$C = \frac{\varepsilon_0 A}{d}$$

To do so for any surface, we

1. Find E (probably using Gauss's law)

2. Find V via integration of E
3. Use $C = Q/V$

Capacitors in parallel:

$$V_{\equiv} = V_1 = V_2 = V_3 = \cdots$$
$$Q_{\equiv} = Q_1 + Q_2 + Q_3 + \cdots$$
$$C_{\equiv} = C_1 + C_2 + C_3 + \cdots$$

Capacitors in series:

$$Q_{\equiv} = Q_1 = Q_2 = Q_3 = \cdots$$
$$V_{\equiv} = V_1 + V_2 + V_3 + \cdots$$
$$\frac{1}{C_{\equiv}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

Energy stored in a capacitor:

$$U = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C}$$

Energy density stored in the electric field:

$$u = \frac{\varepsilon_0 E^2}{2}$$

K is the dielectric constant:

$$C = KC_0$$

Types of Problems

- Deriving the capacitance for a certain kind of capacitor: spherical, cylindrical, parallel plate
- Problems with capacitor circuits
- Problems involving dielectrics
- Problems involving electric energy storage

Current, Resistance, and Circuits

Current:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

Ohm's Law:

$$V = IR$$

Power transformed is:

$$P = IV$$

For a resistor, this is

$$P = I^2 R = \frac{V^2}{R}$$

For resistors in series:

$$V_{\equiv} = V_1 + V_2 + V_3 + \cdots$$
$$I_{\equiv} = I_1 = I_2 = I_3 = \cdots$$

$$R_{\equiv} = R_1 + R_2 + R_3 + \cdots$$

For resistors in parallel:

$$V_{\equiv} = V_1 = V_2 = V_3 = \cdots$$
$$I_{\equiv} = I_1 + I_2 + I_3 + \cdots$$

$$\frac{1}{R_{\equiv}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

At a junction:

$$\sum I = 0$$

Around a loop:

$$\sum \Delta V = 0$$

For adding these changes in a loop:

Element	Direction Moving	ΔV
Resistor	Same as I	$-IR$
Battery	From $-$ to $+$	$+\mathcal{E}$
Capacitor	From $-$ to $+$	$+Q/C$

For charging an RC circuit:

$$\mathcal{E} - R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} = 0$$
$$\Rightarrow Q(t) = C\mathcal{E}\left(1 - e^{-t/(RC)}\right)$$

For discharging an RC circuit:

$$-R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} = 0$$
$$\Rightarrow Q(t) = Q_0 e^{-t/(RC)}$$

Types of Problems

- Circuit problems, esp. involving Kirchoff's rules.
- Write and solve loop and junction equations
- Calculate unknown currents and voltages
- Calculate power delivered or dissipated by circuit elements. EMF sources deliver power to circuits, and resistors dissipate power.
- RC circuit problems. Be able to follow the derivation of charging and discharging.
  - not part of a long question

Magnets

Magnets

Magnets have two poles, north and south. Like poles repel, opposite poles attract.

Furthermore, if you cut a (permanent) magnet in half, it will form smaller submagnets, not an all-north/all-south pair.

Magnetic fields can be visualized using magnetic field lines, which are always closed loops.

The earth is a magnet. The north end of the earth is the south end of a giant weak magnet.

A uniform magnetic field is constant in magnitude and direction. This can be approximately the case often.

Electromagnetism

Electric current can form magnet fields.

Use your right hand, and point your thumb in the direction of the current. The direction your fingers curl is the direction of the magnetic field.

Alternatively, if you have a loop of current, if you curl your fingers in the direction of the current, the magnetic field goes in the direction of your thumb.

Cross product

$$\vec{a} \times \vec{b} = \|a\|\|b\| \sin(\theta)\vec{n}$$

Alternatively,

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Order matters!

Magnetic Force

$$\vec{F} = q\vec{v} \times \vec{B} = qv\sin(\varphi)$$

$\vec{F}$ is the force on a charge q. $\vec{v}$ is the velocity of a charge. $\vec{B}$ is the magnetic field. φ is the angle between $\vec{v}$ and $\vec{B}$.

Note that q is signed.

To remember

- The magnetic force is perpendicular to the direction of the magnetic field.
- The magnetic force is perpendicular to the direction of the velocity.
- Velocity is required
- Right-hand rule is useful

Circle

If a charged particle is moving perpendicular to a uniform magnetic field, its path will be a circle.

$$\|F\| = qvB$$

$$a_c = \frac{v^2}{r} \Rightarrow \frac{\|F\|}{m} = \frac{v^2}{r}$$

Then

$$\frac{qvB}{m} = \frac{v^2}{r}$$
$$\Rightarrow r = \frac{mv}{qB}$$

Example

An electron travels at $v = 1.5 \times 10^7 \text{ m /s}$ in a plane perpendicular to a uniform 0.010 T magnetic field. Describe its path.

$$r = \frac{mv}{qB} = 8.52845 \text{ mm}$$

$$T = \frac{2\pi r}{v} \Rightarrow T = 3.57239 \text{ ns}$$

Review

$3 \to 4$ conceptional problems.

No more RC circuits.

Solutions have been posted for the practice exam.

It's curved to B- usually. It will never be curved down.

Problems

Problem $\vec{E}$ gaussian

A long insulating cylinder of radius a with linear charge density $+\lambda$ has a charge uniformly distributed through its volume. It is surrounded by a conducting cylindrical shell of inner radius b and outer radius c with linear charge density $+2\lambda$.

Charge per volume on the inner cylinder:

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{\lambda L}{L\pi a^2} = \frac{\lambda}{\pi a^2}$$

To find E, consider a cylindrical Gaussian surface with radius r and length L.

$$\Phi_e = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\varepsilon_0}$$
$$E \cdot 2\pi r L = \frac{Q}{\varepsilon_0} E = \frac{Q}{2\varepsilon_0 \pi r L}$$

Let $r < a$.

This means that the Gaussian surface is inside the inner insulating cylinder.

Therefore $Q = \lambda L \frac{\pi r^2}{\pi a^2} = \frac{\lambda r^2}{a^2} L$

$$E = \frac{Q}{2\varepsilon_0 \pi r L} = \frac{\lambda r}{2a^2 \varepsilon_0 \pi}$$

Let $a < r < b$.

Then, $Q = \lambda L$

$$E = \frac{Q}{2\varepsilon_0 \pi r L} = \frac{\lambda}{2\varepsilon_0 \pi r}$$

For $b < r < c$:

$$\vec{E} = 0$$

Inside conductor

For $c < r$:

$Q = (\lambda + 2\lambda)L$

$$E = \frac{3\lambda}{2\varepsilon_0 \pi r}$$

Linear charge density on inner and outer surfaces:

$r = b \to -\lambda$ $r = c \to 3\lambda$

Problem $\vec{E}$

$$\vec{E} = \int \frac{k}{r^2} \hat{r} \, \mathrm{d}q$$

$\mathrm{d}q = \frac{Q}{L} \, \mathrm{d}x$

$$\begin{aligned}
&= \int_0^L \frac{k}{r^2} \hat{r} \frac{Q}{L} \, \mathrm{d}x \\
&= \frac{kQ}{L} \int_0^L \frac{1}{r^2} \hat{r} \, \mathrm{d}x \\
&= \frac{kQ}{L} \int_0^L \frac{1}{\sqrt{x^2 + a^2}^2} \underbrace{\frac{x\hat{\imath} + a\hat{\jmath}}{\sqrt{x^2 + a^2}}}_{\hat{r}} \, \mathrm{d}x \\
&= \frac{kQ}{L} \left(\int_0^L \frac{x\hat{\imath}}{\sqrt{x^2 + a^2}^3} + \int_0^L \frac{a\hat{\jmath}}{\sqrt{x^2 + a^2}^3} \, \mathrm{d}x \right)
\end{aligned}$$

Helical Motion

Recall from last time we calculated what happens if a particle moves in a circle and what would cause this (velocity perpendicular to the field).

Consider what would happen if you add another velocity parallel to the field. The velocity parallel does not change $q\vec{v} \times \vec{B}$ because it is parallel, but the velocity perpendicular still has an effect, and so the acceleration due to the magnetic field is 0 in the direction of the magnetic field and creates a circle in the perpendicular direction. In sum, this makes the particle go in a helix.

Therefore, a helix is the general form of the movement of a particle in a magnetic field of constant direction and magnitude.

- If the particle has velocity components parallel to and perpendicular to the field, its path is a helix.

- The speed and kinetic energy of the particle remains constant.
- The perpendicular component of the velocity, $v_\perp$, determines the circular part of the motion
- The parallel component, $v_\parallel$, determines the translational part of the motion

Force on a conductor

The force on a segment of straight wire is

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

Where I is the current, $\vec{\ell}$ is the vector length of the wire, $\vec{B}$ is the magnetic field and $\vec{F}$ is the force.

Torque on a conductor

By using the above statement, we can also find the torque on a wire.

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

μ is the magnetic dipole moment and is calculated $\vec{\mu} = NI\vec{A}$, where $\vec{A}$ is the area and is in the direction perpendicular to the face, N is the number of loops of wire, and I is the current in the wire.

This has practical applications too. Using this torque, you can create an electric motor by turning off power to an electromagnetic at specific times.

Velocity Selector

By using the fact that magnetic fields and electric fields both act on a particle and that magnetic field's strength is dependent on velocity, you can create a velocity selector for charged particles.

For the particle to remain going straight, the forces must sum to zero:

$$\begin{aligned} F_B + F_E &= 0 \\ qvB + qE &= 0 \\ q(vB + E) &= 0 \\ vB + E &= 0 \\ vB &= -E \\ v &= -\frac{E}{B} \end{aligned}$$

Therefore I had a sign error somewhere (I think the slides are wrong), which has a velocity requirement of $v = \frac{E}{B}$ for the particle to get through, assuming the selector is long enough to detect any acceleration by having the particle hit the walls.

This can then be used to create a mass spectrometer by first ensuring that particles have a certain velocity and then measuring how far a magnet displaces them. Since the magnetic force does not depend on the mass, the acceleration depends on the mass. Now, you can solve for the mass since you have an exact starting velocity and position.

Hall effect

Since magnets affect all charged particles, this includes particles that are traveling in conductors and or in some solution.

Take a wire where charged particles are moving from left to right. Putting a magnetic field (into the page) through this current will induce a voltage difference between the top and bottom of the wire.

If this charged particle is negatively charged, the bottom will be at a higher potential. Conversely, if this charged particle is positively charged, the bottom will be at a lower potential.

This electric field produced by this movement is called the hall field and is denoted E_H. The magnitude of the potential difference is called the hall emf.

Review

Magnetic force on a moving charge

$$\vec{F} = q\vec{v} \times \vec{B}$$

Magnetic force on a current-carrying segment of wire:

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell} \times \vec{B}$$

If the field is constant and the wire is straight

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

Torque and dipole moment:

$$\vec{\mu} = NI\vec{A}, \qquad \vec{\tau} = \vec{\mu} \times \vec{B}$$

Velocity selector

$$v = \frac{E}{B}$$

Biot-Savart Law

The magnetic field from a small, current-carrying wire segment at a point p, r distance away from the wire segment.

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

$\mathrm{d}\vec{B}$ is the magnetic field at point P induced by the segment

I is the current in the segment.

$\mathrm{d}\vec{\ell}$ is in the direction of the current, and has length equal to the length of the wire segment.

$\hat{r}$ is a unit vector from the segment to point p.

r is the distance from the wire segment to the point p

μ_0 is the magnetic constant in N^2/A

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

where c is the speed of light.

Example

A small line segment carries a current I in the vertical direction. What is the magnetic field at a distance x from the segment?

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

$\mathrm{d}\vec{\ell} \times \hat{r}$ is in the $-\hat{k}$ direction.

Note $\sin(\pi - \theta) = \sin(\theta)$

$$
\begin{aligned}
B &= \frac{\mu_0 I}{4\pi} \int \frac{\sin(\theta)}{x^2 + \ell^2} \,\mathrm{d}\ell \\
&= \frac{\mu_0 I}{4\pi} \int \frac{1}{x^2 + \ell^2} \frac{x}{\sqrt{x^2 + \ell^2}} \,\mathrm{d}\ell \\
&= \frac{\mu_0 I}{4\pi} \int \frac{x}{(x^2 + \ell^2)^{\frac{3}{2}}} \,\mathrm{d}\ell \\
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{x}{(x^2 + \ell^2)^{\frac{3}{2}}} \,\mathrm{d}\ell \\
&= \frac{\mu_0 I x}{4\pi} \frac{2}{x^2} \\
&= \frac{\mu_0 I}{2\pi x}
\end{aligned}
$$

Add on the direction for the vector:

$$\vec{B} = -\frac{\mu_0 I}{2\pi x} \hat{k}$$

Therefore, for a straight wire (of infinite length), the magnetic field strength is inversely proportional to the distance from a wire.

Force between two parallel wires

The magnetic field produced at the position of wire 2 due to the current in wire 1 is:

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

The direction of this is into the page.

$$F_2 = I_2 \ell_2 B_1$$

$$F_2 = \frac{\mu_0 I_1 I_2 \ell_2}{2\pi d}$$

Since F_2 is in the direction of $\vec{l} \times \vec{B}$, it is towards wire 1.

Force on a loop of current

Remembering $\vec{l} \times \vec{B}$, and that $\vec{B}$ from an infinite wire goes into the page, we can break this problem into segments.

$$
\begin{aligned}
F_1 &= \frac{I_2 q \mu_0 I_1}{2\pi b} \left(-\hat{i}\right) \\
F_3 &= \frac{I_2 q \mu_0 I_1}{2\pi 2b} \left(\hat{i}\right)
\end{aligned}
$$

The force on the top and bottom is:

$$F_2 + F_4$$

F_2's direction is in the $\hat{j}$ direction, and F_4's direciton is in the $-\hat{j}$ direction. When summing these forces, the total force is zero.

Then, sum the F_1 and F_3, resulting in

$$-\frac{\mu_0 I_1 I_2 a}{4\pi b}\hat{i}$$

What is F_2 anyway, though?

$$\vec{B}_1(x) = \frac{\mu I_1}{2\pi x}\left(-\hat{k}\right)$$

$$\begin{aligned} F_2 &= \int_b^{2b} \frac{\mu I_2}{2\pi x}\left(-\hat{k}\right) \times \left(\hat{i}\right) \mathrm{d}x \\ &= \int_b^{2b} \frac{\mu I_2}{2\pi x}\hat{j} \,\mathrm{d}x \\ &= \frac{\mu I_2}{2\pi}\hat{j} \int_b^{2b} \frac{1}{x} \,\mathrm{d}x \\ &= \frac{\mu I_2}{2\pi}\hat{j} \ln(2) \end{aligned}$$

Example
Determine $\vec{B}$ at point C in terms of R_1, R_2, θ and the current I.

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

Along the top arc:

$$\vec{B} = \frac{\mu_0 I}{4\pi R_2^2}\underbrace{(R_2\theta)}_{\ell}\left(-\hat{k}\right) = \frac{\mu_0 I \theta}{4\pi R_2}\left(-\hat{k}\right)$$

The bottom arc is very similar for the top arc, but it goes in the opposite direction and uses the smaller radius R_1:

$$\vec{B} = \frac{\mu_0 I \theta}{4\pi R_1}\left(\hat{k}\right)$$

The two side parts will cancel as I is going in opposite directions on them.

Important Equations

Biot-Savart Law

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

This is the analogue to the electric field for a point charge.

Magnetic Field due to current in a long, straight wire

$$B = \frac{\mu_0 I}{2\pi r}$$

Force on wire 2 due to the field from wire 1 for parallel wires:

$$F_2 = \frac{\mu_0 I_1 I_2 \ell_2}{2\pi d}$$

Ampere's Law

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enclosed}}$$

This relates the magnetic field around a closed loop to the total current flowing through the loop.

Example

Use Ampere's Law to find the field around a long, straight wire.

Use a circular path with the wire at the center:

$\vec{B}$ is tangent to $\mathrm{d}\vec{\ell}$ along the path.

Furthermore, B is constant on the path:

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I$$

$$\vec{B} \cdot \oint \mathrm{d}\vec{\ell} = \mu_0 I$$

$$B 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Which is the same as what we previously got with the Biot-Savart law.

When to use

- It is usually only useful with problems with lots of symmetry, where you can use symmetry to identify the direction of the magnetic field.
- Choose an integration path that reflects the symmetry (typically, the path is along lines where the field is constant and perpendicular to the field where it is changing).
- Use the enclosed current to determine the field
- Similar to Gauss's law problems, but it's a line integral, not a surface integral.

Example

A coaxial cable consists of a solid inner conductor of radius R_1, surrounded by a concentric cylindrical tube of inner radius R_2 and outer radius R_3. The conductors carry equal and opposite currents of I_0 distributed uniformly across their cross sections. Determine the magnetic field at a distance r from the axis for

1. $r < R_1$
2. $R_1 < r < R_2$
3. $R_2 < r < R_3$
4. $r > R_3$

Solution

For $r < R_1$:

$$B(2\pi r) = \mu_0 I_0 \underbrace{\frac{\pi r^2}{\pi R_1^2}}_{\text{portion of area}}$$

$$B = \frac{\mu_0 I_0 r}{2\pi R_1^2}$$

By the right-hand rule, since $\vec{B} = \ell \times \hat{r}$, the direction of $\vec{B}$ is counterclockwise.

For $R_1 < r < R_2$:

Reduces to the normal wire case: $B = \frac{\mu_0 I}{2\pi r}$

For $R_2 < r < R_3$:

$$B(2\pi r) = \mu_0 \left(I_0 - I_0 \frac{\pi(r^2 - R_2^2)}{\pi(R_3^2 - R_2^2)} \right)$$

$$B = \frac{\mu_0 I_0}{2\pi r} \left(1 - \frac{r^2 - R_2^2}{R_3^2 - R_2^2} \right)$$

$$B = \frac{\mu_0 I_0}{2\pi r} \left(\frac{R_3^2 - R_2^2}{R_3^2 - R_2^2} - \frac{r^2 - R_2^2}{R_3^2 - R_2^2} \right)$$

$$B = \frac{\mu_0 I_0}{2\pi r} \left(\frac{R_3^2 - R_2^2 - (r^2 - R_2^2)}{R_3^2 - R_2^2} \right)$$

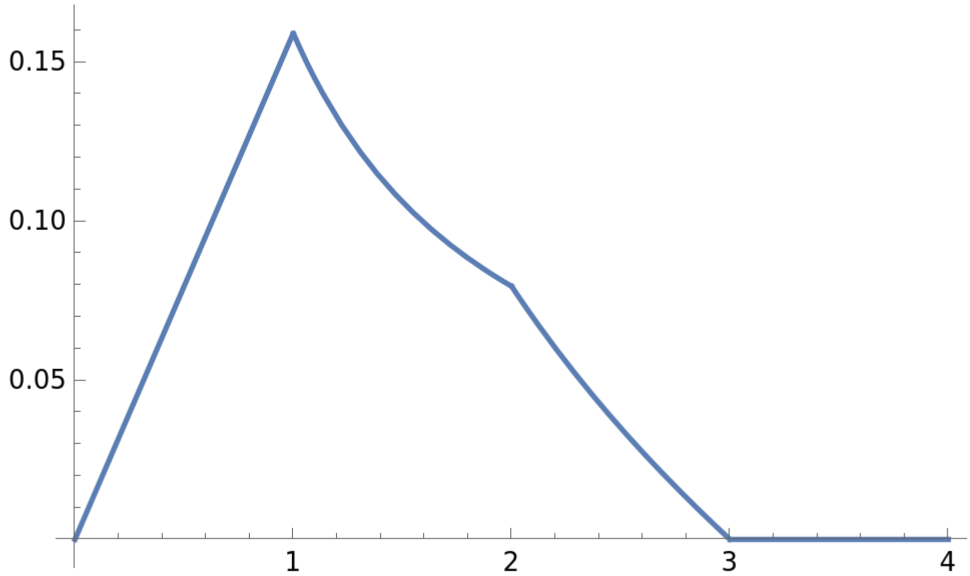
$$B = \frac{\mu_0 I_0}{2\pi r} \left(\frac{R_3^2 - r^2}{R_3^2 - R_2^2} \right)$$

For $r > R_3$

$$B(2\pi r) = \mu_0 (I_0 - I_0)$$

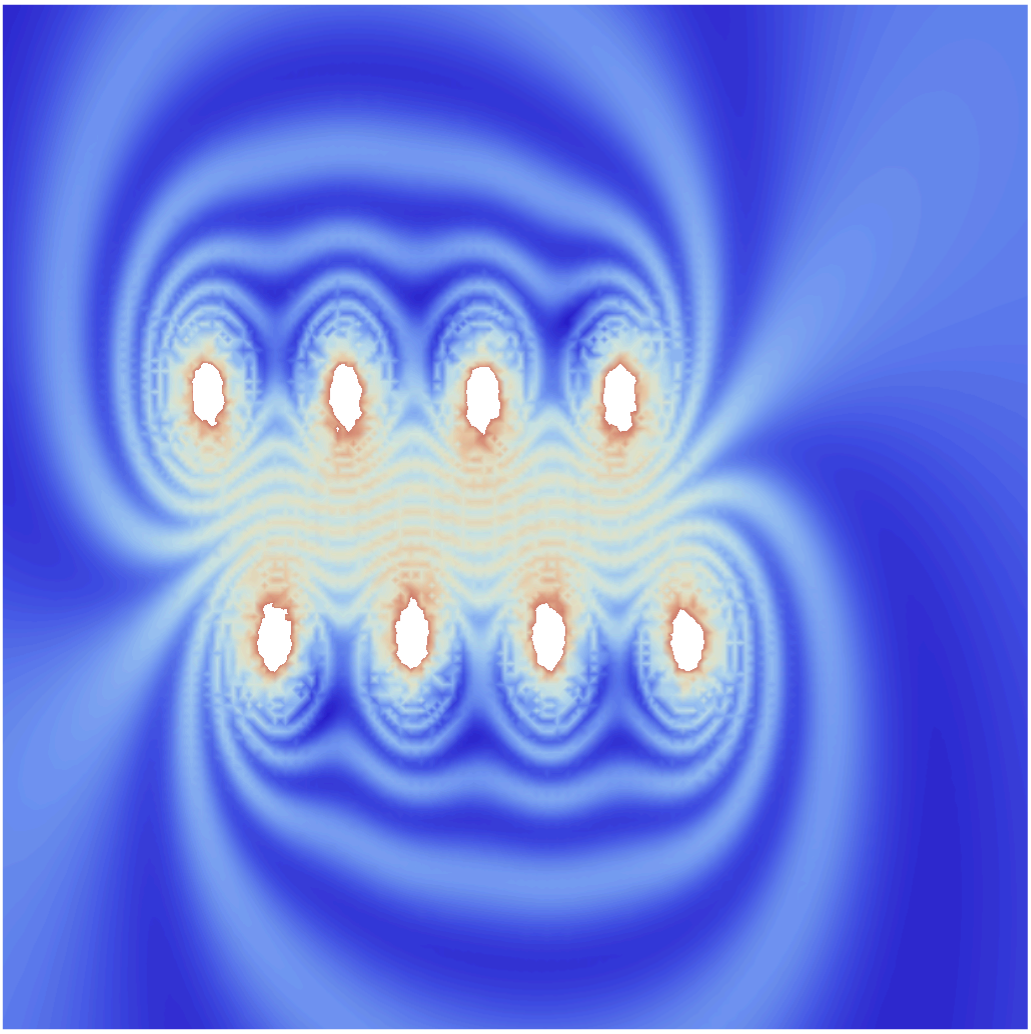
$$B(2\pi r) = \mu_0 \cdot 0$$

$$B = 0$$



Solenoid

A solenoid is many loops of wire packed very close together.



Take a coil of wire with 4 loops. Above is the density plot of the magnetic field. Notice that the field “mostly” cancels outside of the center, and inside the solenoid, the field is much stronger.

To compute the field inside the solenoid, draw a rectangle with corners a, b, c, d, such that cb and da are long enough such that ab has zero magnetic field. cb and da are perpendicular to the magnetic field, so they have zero contribution to the ampere’s law path integral. Let the length of ab and of cd be ℓ. Then:

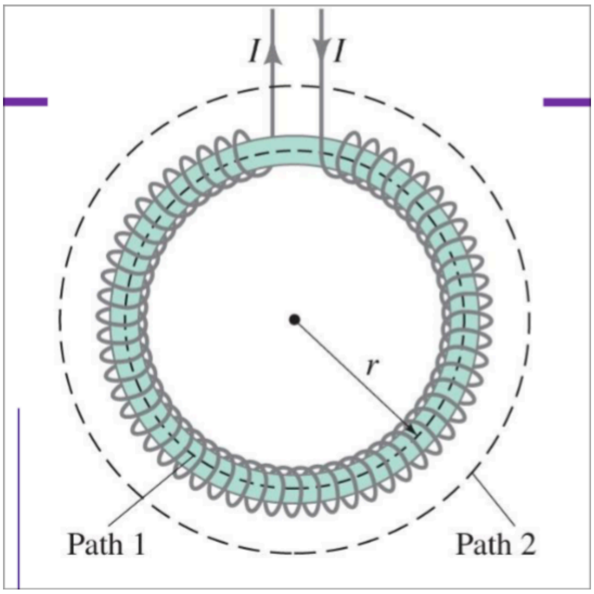
$$\ell B_{\text{inside}} = \mu I_{\text{enclosed}} = \mu N I$$

$$B_{\text{inside}} = \frac{\mu_0 N I}{\ell} = \mu_0 n I$$

Where I is the current in the solenoid wire, N is the number of loops enclosed, and n is the number of loops per unit length.

Toriod

Use Ampère’s law to determine the magnetic field inside and outside a toroid, which is like a solenoid with current I, bent into the shape of a circle as shown with N loops.



$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enclosed}}$$
$$B(2\pi r) = \mu_0 NI$$
$$B_{\text{inside}} = \frac{\mu_0 NI}{2\pi r}$$
$$I_{\text{enclosed}} = 0 \to B_{\text{outside}} = 0$$

Magnetic Field of a Moving Charge

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

$\hat{r}$ is the unit vector from the point charge toward where the field was measured.

Induction Experiment

A coil of wire was placed down and connected to an ammeter. Moving a magnet up and down near the coil shows a current in the coil!

This magnet could also be an electromagnet, but it still requires movement in the electromagnet.

However, you can also just turn on and off the electromagnet, and this also works.

Magnetic Flux

$$\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$$

This is analogous to the calculation of electric flux.

If $\vec{A}$ and $\vec{B}$ are parallel, $\Phi_B = BA$.

If $\vec{A}$ and $\vec{B}$ are perpendicular, $\Phi_B = 0$

The unit of magnetic flux is the webber: $\mathrm{Wb} = \mathrm{V} \cdot \mathrm{s} = \mathrm{T} \cdot \mathrm{m}^2$

Faraday's Law

When the magnetic flux through a single closed loop changes with time, there is an induced emf that can drive a current around the loop:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

For Faraday's law, put your thumb in the direction of decreasing flux, and then your fingers curl in the $+\mathcal{E}$ direction.

Solenoid

As for magnitude, if you stretch out a circle of wires that was previously a circle into an ellipse, you reduce the area and therefore reduce the magnetic flux (in magnitude, assuming the magnetic field is constant). This then creates a current.

Lenz's Law

The direction of any magnetic induction effect is such to oppose the cause of the effect.

For example, if you move a magnet (north down) toward a loop, since like poles repel, and the "goal" of the loop is to have the magnet not move, it will create a north field in response.

Equations

Magnetic flux

$$\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$$

Faraday's Law

$$\mathcal{E} = -N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

General form

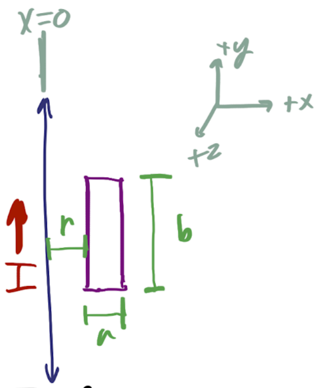
$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

Example 1

A flat rectangular wire loop with width a and height b is positioned next to a long straight current-carrying wire. Both the loop and the wire are in the plane of the page, and the direction of the current is clearly indicated in the figure.

What is the flux through the loop when it is a distance r from the wire?

If the loop has total resistance R and is being pulled to the right at speed v, what is the induced current in it? What is its direction?



Magnetic Field

$$\vec{B}(x) = \frac{\mu_0 I}{2\pi x}(-\hat{k})$$

Magnetic Flux

$$dA = b\,dx$$

$$\Phi_B = \int_{x=r}^{x=r+a} B(x)\, b\, dx = -\frac{\mu_0 I b}{2\pi} \log\left(\frac{r+a}{r}\right)$$

Emf

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d\Phi_B}{dr}\underbrace{\frac{dr}{dt}}_{v} =$$

$$\frac{d\Phi_B}{dr} = -\frac{abI\mu_0}{2\pi r(a+r)} \to$$

$$\mathcal{E} = \frac{\mu_0 abIv}{2\pi r(a+r)} \to I = \frac{\mathcal{E}}{R} = \frac{\mu_0 abIv}{2\pi r R(a+r)}$$

Example 2

A conducting rod is moved at velocity v on top of a U-shaped conductor in a constant magnetic field going through the U-shaped conductor.

$$\mathcal{E} = \frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

$$\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$$

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = B\frac{\mathrm{d}A}{\mathrm{d}t} = B\frac{\mathrm{d}}{\mathrm{d}t}[\ell \times x] = B\ell v$$

$$\mathcal{E} = B\ell v$$

Example 3

The rod shown below moves to the right on essentially zero-resistance rails at a speed $v = 3.0\ \mathrm{m/s}$, where the rails are away from each other at a distance of $4\ \mathrm{cm}$. If the magnetic field is $B = 0.75\ \ \mathrm{T}$ everywhere in the region, what is the current through the resistor $(5\ \Omega)$? Does the current circulate clockwise or counterclockwise?

For direction, there is increasing flux out of the page, which implies the current is clockwise to counter the increasing flux.

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}, \Phi_B = BA$$

$$\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = B\frac{\mathrm{d}A}{\mathrm{d}t} = Bhv$$

$$I = \frac{\mathcal{E}}{R} = \frac{Bhv}{R} \to I = 0.018\ \text{A}$$

EMF induced in a moving conductor

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

This is upward. This will cause a charge separation, which creates an upward electric field, due to the downward force on the electron.

Equilibrium occurs when $\vec{F}_B + \vec{F}_E = 0$.

$$qE_{\text{induced}} = qvB \to E_{\text{induced}} = vB$$

For EMF this means that

$$\mathcal{E} = E_{\text{induced}}\ell = vB\ell`$$

Generalized Form of Faraday's Law

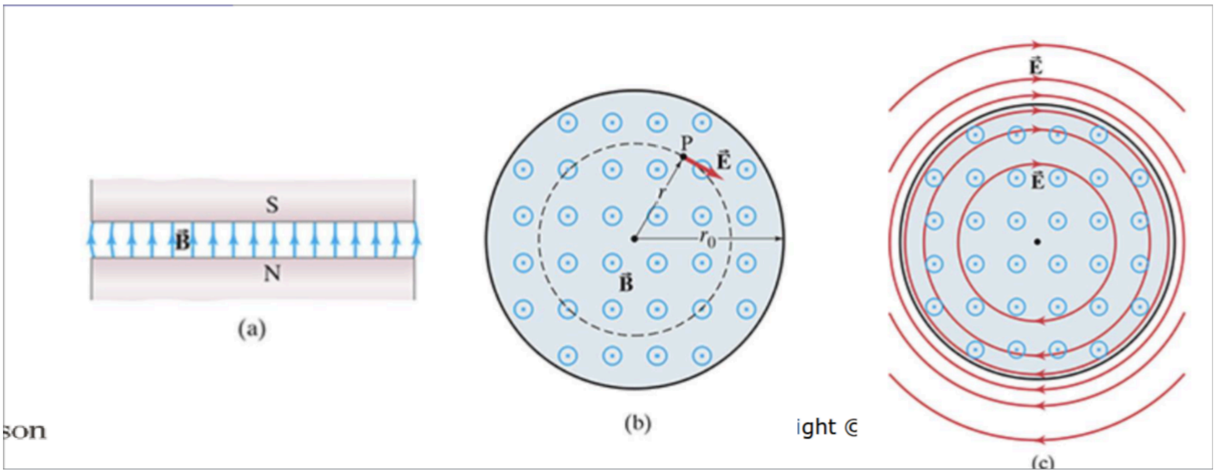
A changing magnetic flux induces an electric field, which is a generalization of Faraday's law. This electric field will *always* exist, regardless of whether or not there are conductors around to carry current.

$$\oint \vec{E} \cdot \mathrm{d}\ell = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

With statics, this was zero. No longer are we static, though: the integral around a closed loop depends on how the magnetic flux through the loop is *changing*.

Example

A magnetic field between the pole faces of an electromagnet is nearly uniform at any instant over a circular area of radius, as shown in the figures. The current in the windings of the electromagnet is increasing in time at a constant rate α at each point. Beyond the circular region ($r > r_0$), we assume $B = 0$ at all times. Determine the electric field at any point P a distance r from the center of the circular area due to the changing magnetic field.



For $r < r_0$.

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \alpha > 0$$

B is constant in space but varying in time.

We may not have a wire for there to be a current, but if there was one, I would be to counter the changing flux, and therefore E is such to be compatible. Therefore, E is tangent to the loop and goes clockwise. By symmetry, E is constant around the loop.

$$\int \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$
$$E(2\pi r) = -\frac{\mathrm{d}(BA)}{\mathrm{d}t}$$
$$E(2\pi r) = -\pi r^2 \frac{\mathrm{d}B}{\mathrm{d}t}$$
$$E = -\frac{r}{2}\frac{\mathrm{d}B}{\mathrm{d}t}$$
$$E = -\frac{r\alpha}{2}$$

For $r > r_0$

$$E(2\pi r) = \pi r_0^2 \frac{\mathrm{d}B}{\mathrm{d}t}$$
$$E = \frac{r_0^2 \alpha}{2r}$$

Electric Generators

If there is a loop rotating between two magnets, rotating with constant angular velocity ω, the induced emf is sinusoidal:

$$\mathcal{E}(t) = -BA\omega \sin(\omega t)$$

For N loops, $\mathcal{E}_0 = NBA\omega$ and $\mathcal{E}(t) = \mathcal{E}_0 \sin(\omega t)$.

Mutual Inductance

Consider two neighboring coils of wire.

If the current in coil 1 changes, this will induce emf in coil 2, and if the current in coil 2 changes, this will induce emf into coil 1.

The proportionality constant for this pair of coils is called the mutual inductance, M.

Derivation

Let coil 1 have N_1 turns and let coil 2 have N_2 turns.

$$\Phi_{21} = \int \vec{B}_i \, \mathrm{d}\vec{A}$$

We know, therefore, $B_1 \propto I_1$, and furthermore

$$\frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t} \propto$$

Furthermore, mutual inductance is symmetric: $M = M_{21} = M_{12}$

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}$$

The unit of inductance is the henry: $1 \text{ H } = 1 \text{ Wb/A }$.

Application

Electric toothbrushes charge via mutual inductance: the base has alternating current, which induces emf in a coil in the toothbrush.

Example

A long thin solenoid of length ℓ and cross-sectional area A_1 contains N_1 closely packed turns of wire. A coil of N_2 turns is wrapped around it. Assume all the flux from coil 1, the solenoid passes through coil 2, and calculate the mutual inductance.

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}$$

Since field B_1 is known, we shall use that:

$$B_1 = \mu_0 n I_1 = \mu_0 \frac{N_1}{\ell} I_1$$

and therefore

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_2 B_1 A}{I_1} = \frac{N_2 \mu_0 \frac{N_1}{\ell} I_1 A_1}{I_1} = \frac{\mu_0 N_1 N_2 A_1}{\ell}$$

Other Question

What would create the most inductance between 2 flat circular coils?

Having them face to face! This is because having them face-to-face puts as much flux through the other coil as possible.

Self-Inductance

Any circuit with a coil that carries a varying current I has a self-induced emf.

$$L = \frac{N \Phi_B}{I}$$

L is the self-inductance. N is the number of turns. Φ_B is the flux due to the current through each turn of the coil.

You generally ignore the single loops that just happen to exist in circuits because they are usually irrelevant.

Example

Determine a formula for the inductance of a solenoid with N loops and length ℓ whose cross-sectional area is A.

Inside the solenoid

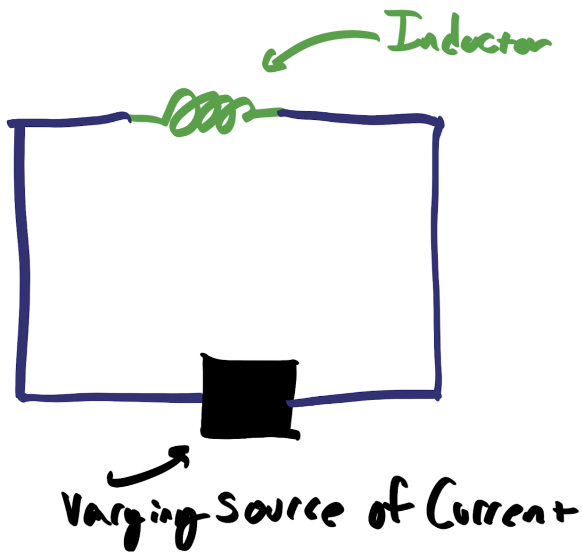
$$B = \mu_0 n I = \mu_0 \frac{N}{\ell} I$$

Therefore

$$L = \frac{N \Phi_B}{I} = \frac{N \mu_0 \frac{N}{\ell} I A}{I} = \frac{\mu_0 N^2 A}{\ell}$$

Inductors as circuit elements

We have a voltage source that gives a variable amount of $\mathcal{E}$.



Over an inductor, $\Delta V = -L\frac{\mathrm{d}I}{\mathrm{d}t}$

If you have no varying current, there is no potential difference.

If you do have a varying current, emf is generated in the reverse direction of the change in current. The magnitude of this change depends on the inductance of the inductor and the size of the change in current.

Unlike resistors, inductors do not delete energy, they just store the energy to be released later:

$$U = L\int_0^I i\,\mathrm{d}i = \frac{1}{2}LI^2$$

The energy density is therefore

$$u = \frac{U}{V} = \frac{\frac{1}{2}LI^2}{V} = \frac{(\mu_0 N^2 A I^2)/(2\ell)}{A\ell} = \frac{1}{2}\mu_0 N^2 A I^2$$

Furthermore, since

$$B = \mu_0 \frac{N}{\ell} I \to I = \frac{\ell B}{\mu_0 N}$$

$$u = \frac{1}{2}\mu_0 N^2 A \left(\frac{\ell B}{\mu_0 N}\right)^2 =$$

(something went wrong above)

$$\mu = \frac{B^2}{2\mu_0}$$

But the above is in a vacuum, if it is not in a vacuum, $\mu = K_m \mu_0$ and

$$\mu = \frac{B^2}{2\mu}$$

R-L circuit

An R-L circuit contains a resistor, an inductor and maybe an emf source.

Suppose at some time $t = 0$ you close a switch, connecting the emf source to the resistor and inductor. The current does not instantly change, and will take a while to get to the steady state current $I_0 = \frac{V_0}{R}$, governed by the time constant $\tau = L/R$.

Solving

By Kirchoff's loop law:

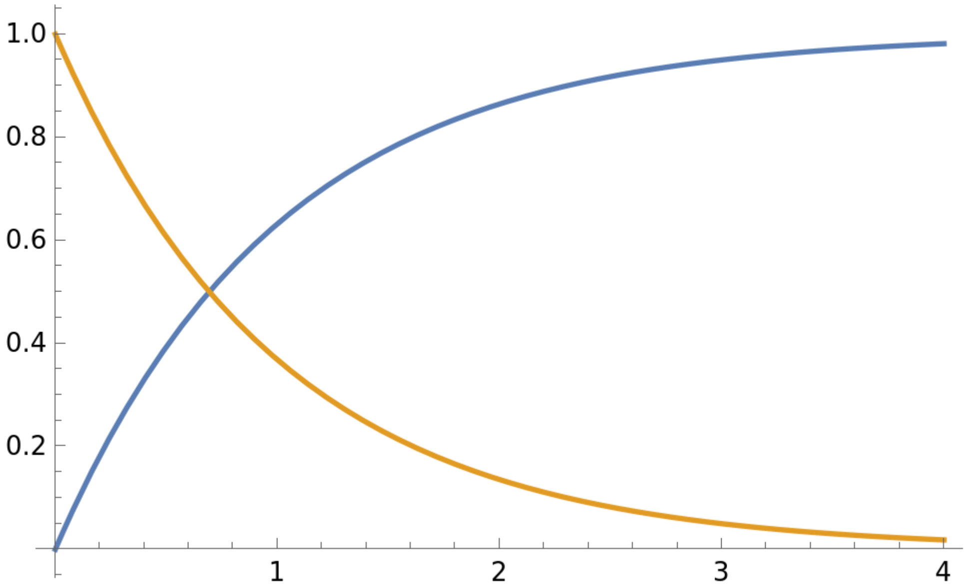
$$+\mathcal{E} - IR - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

Solve with initial condition $I(0) = 0$.

$$\begin{aligned} I(t) &= \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right) \\ &= I_0\left(1 - e^{-t/\tau}\right) \end{aligned}$$

Similarly, for current decay, with initial condition $I(0) = I_0$:

$$\begin{aligned} I(t) &= \frac{\mathcal{E}}{r}e^{-Rt/L} \\ &= I_0 e^{-t/\tau} \end{aligned}$$



Plot with $I_0 = 1$ and $\tau = 1$. The current decay is in orange, and the current growth is in blue.

L-C circuit

It oscillates!

Start with a capacitor charged with a potential difference V_m and initial charge $Q_m = CV_m$ on its left-hand plate.

Connect the capacitor to an inductor.

The capacitor discharges through the inductor. As it discharges, the current continues to flow, but it will slow down, which means that $\frac{\mathrm{d}I}{\mathrm{d}t}$ is negative

When the potential across the capacitor becomes zero, emf is zero and the current levels off at it's maximum value I_m.

... see slides

Derivation

$$+\frac{q}{C} - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

But q also depends on I:

$$I = -\frac{\mathrm{d}q}{\mathrm{d}t} \to \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

$$+\frac{q}{C} - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$$

$$\frac{q}{C} + L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 0$$

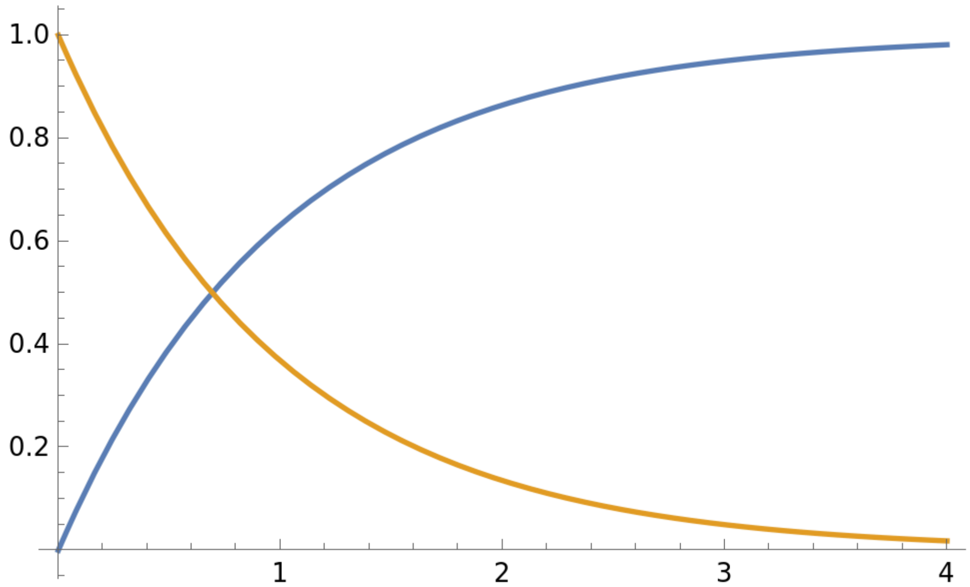
Solving:

$$q(t) = \mathbb{c}_1 \cos\left(\frac{t}{\sqrt{LC}} + \mathbb{c}_2\right)$$

Therefore:

$$I(t) = \mathbb{c}_1 \frac{1}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}} + \mathbb{c}_2\right)$$

Recap: Current in R-L circuit



Plot with $I_0 = 1$ and $\tau = 1$. The current decay is in orange, and the current growth is in blue.

Current decay is $I(t) = I_0 e^{-t/\tau}$ and current growth is $I(t) = I_0\left(1 - e^{-t/\tau}\right)$. $\tau = L/R$ and $I_0 = \mathcal{E}/R$.

Recap: Current in R-C circuit

$$I(t) = I_0 \sin(\omega t + \varphi)$$
$$I_0 = \omega Q_0$$
$$\omega = \frac{1}{\sqrt{LC}}$$
$$q(t) = Q_0 \cos(\omega t + \varphi)$$

Q_0 is the maximum charge on the capacitor. If this is at time $t = 0$, $\varphi = 0$.

Therefore $\omega = \frac{1}{\sqrt{LC}}$.

The period is $T = \frac{2\pi}{\omega}$. Using the previous formula, where $\varphi = 0$, this can get very nice values at values $t \in \{0, T/4, T/2, 3T/4, T\}$.

Energy

$$q = Q_0 \cos(\omega t)$$
$$I = \omega Q_0 \sin(\omega t)$$

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q_0^2}{C}\cos^2(\omega t)$$
$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}L\omega^2 Q_0^2 \sin^2(\omega t)$$

$$\omega^2 = \frac{1}{LC} \Rightarrow L = \frac{1}{c\omega^2}$$
$$U_B = \frac{1}{2}\frac{Q_0^2}{C}\sin^2(\omega t)$$

Therefore

$$U = U_E + U_B = \frac{1}{2}\frac{Q_0^2}{C}(\cos^2(\omega t) + \sin^2(\omega t))$$
$$= \frac{1}{2}\frac{Q_0^2}{C}$$

This is constant.

In general, the inductor-capacitor circuit is very similar to the mass-spring system from mechanics.

RLC circuit

Consider a circuit that starts with a fully charged capacitor.

$$0 = -L\frac{\mathrm{d}I}{\mathrm{d}t} - IR + \frac{q}{C}$$
$$I = -\frac{\mathrm{d}q}{\mathrm{d}t}$$

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = 0$$

These are difficult to solve, but it is possible.

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$Q = Q_0 e^{\frac{R}{2L}t} \cos(\omega' t + \varphi)$$

AC current

$$V = V_0 \sin(\omega t)$$

$$I = I_0 \sin(\omega t)$$

To represent these vectors, we define phasors, rotating vectors. By projecting them onto the x -axis, you get the current.

The average voltage and current is 0, but that isn't useful. To get a sense of their typical values, they are squared, averaged, and then square-rooted, yielding the RMS (root-mean-square) value.

$$I = I_0 \cos(\omega t)$$

$$\text{RMS} = \frac{I_0}{\sqrt{2}}$$

Resistors

Assume $I = I_0 \cos(\omega t)$. By ohm's law,

$$V = IR = I_0 R \cos(\omega t)$$

$$V = V_0 \cos(\omega t)$$

.

Therefore, the voltage is in phase with the current for the resistor.

Inductor

Assume $I = I_0 \cos(\omega t)$.

Then

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} = -\omega I_0 L \sin(\omega t)$$

$$V = V_0 \qquad \cos\left(\omega t + \frac{\pi}{2}\right)$$

And furthermore $V_0 = \omega L I_0$.

Notice that the current is lagging the voltage by $90°$.

$V_0 = I_0 X_L$. X_L is called the inductive reactance and is in units of ohms.

$$X_L = \omega L = 2\pi f \omega$$

Capacitor

$I = I_0 \cos(\omega t)$

Find the voltage across the capacitor.

$$V = \frac{Q}{C}, Q = \int_0^t I(t')\mathrm{d}t' = \frac{I_0}{\omega}\sin(\omega t)$$

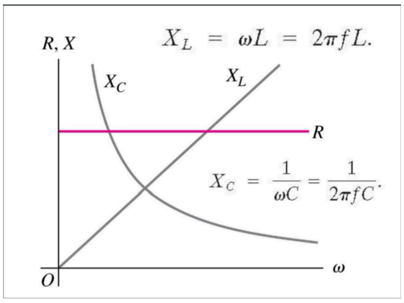
$$V = V_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

And $V_0 = \frac{I_0}{\omega C}$

X_C is called the capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

Comparison



R is independent of frequency. For an inductor, as $\omega \to \infty$, the current becomes smaller and smaller. For a capacitor, and $\omega \to 0$, there is no current.

For a combo RLC AC circuit, things get complicated.

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

But:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0}$$

The current is the same through all the elements, and the voltage always adds.

After some time, the phase portrait has rotated by ω. But all the angles between the components are maintained, so all you have to do is project the voltages down to the x-axis.

Magnetic Force

For a moving charge:

$$\vec{F} = q\vec{v} \times \vec{B}$$

For a straight wire segment with current:

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

This can be made infinitesimal:

$$\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{\ell} \times \vec{B}$$

Problems

- Magnetic field acting on a moving charge
  - Circular motion in a uniform charge
  - Velocity selector/mass spectrometer
  - Helical motion

- Magnetic field acting on a current-carrying wire
  - magnetic force on a floating wire
  - force on a rectangular loop due to magnetic field
  - force on current carrying wire

Biot-Savart Law

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

Ampere's Law

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enclosed}}$$

This is an integral around the edge of some closed loop.

This implies

$$B = \frac{\mu_0 I}{2\pi r}$$

For the magnetic field due to current in a straight wire.

For a solenoid

$$B_{\text{inside}} = \mu_0 n I$$
$$B_{\text{outside}} = 0$$

n is the number of wraps per unit length.

Problems

- Using $B = \frac{\mu_0}{2\pi r}$
  - force between 2 parallel wires
  - force on a loop of current due to a field from a long straight wire
- Using Biot-Savart law
  - Deriving magnetic field for long, straight wire
- Ampere's law
  - Calculate the magnetic field for when you have symmetry

Faraday's Law of Induction

Magnetic flux is

$$\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$$

The emf is then

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

The general form is then

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

Problems

- Faraday's Law
  - Induced emf in a square loop pulled away from a wire
  - Induced emf in a loop with a changing area
  - Loop moving in a magnetic field
  - Changing magnetic field induces current

...more

Inductance

$$\mathcal{E}_2 = -N_2 \frac{\mathrm{d}\Phi_{21}}{\mathrm{d}t}$$

Φ_{21} is the flux through coil 2 due to the field produced by coil 1. Then, mutual inductance is:

$$\mathcal{E}_2 = -M_{21} \frac{\mathrm{d}I_1}{\mathrm{d}t}$$

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}$$

Self-inductance is:

$$L = \frac{N \Phi_B}{I}$$

Then, $\mathcal{E} = -L \frac{\mathrm{d}I}{\mathrm{d}t}$.

Notice how resistors disappear energy, but inductors *do not*. They just store the current in the magnetic field, to be used later (when the current decreases again to zero).

Then this implies the magnetic energy density:

$$u = \frac{B^2}{2\mu_0}$$

Inductors, as a circuit element, are used to oppose sudden changes in current by storing energy in the magnetic field.

Circuits with inductors

R-L circuit: Current decay is $I(t) = I_0 e^{-t/\tau}$ and current growth is $I(t) = I_0\left(1 - e^{-t/\tau}\right)$. $\tau = L/R$ and $I_0 = \mathcal{E}/R$.

R-C circuit: $I(t) = I_0 \sin(\omega t + \varphi)$, $I_0 = \omega Q_0$, $\omega = \frac{1}{\sqrt{LC}}$, $q(t) = Q_0 \cos(\omega t + \varphi)$

RLC circuits will not be on the exam, but they do have an analytic solution.

Problems

- Mutual inductance
  - Solenoid and coil
  - Wire and rectangular loop
- Self-inductance
  - Solenoid

Displacement Current

Ampere's Law relates a line integral to the current through a surface. This is for any surface, not just a flat one.

Ampere's law's is incomplete, as can be shown by considering the process of charging a capacitor.

When a capacitor charges, current exists to charge the capacitor. But if you change your surface to go through the space between the capacitor plates, you get zero, a different result, which is a contradiction.

This is corrected with displacement current:

$$I_D = \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$$

To fully correct Ampere's Law:

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \left(I_{\text{enclosed}} + \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \right)$$

The above is for $\varepsilon = \varepsilon_0$.

Example

Suppose that a circular parallel-plate capacitor has radius R and plate separation d. A sinusoidal potential difference $V = V_0 \sin(\omega t)$ is applied across the plates.

1. In the region between the plates, show that the magnitude of the induced magnetic field is given by $B = B_0(r)\cos(\omega t)$ where $B = B_0(r)$ is a function of the radial distance r from the capacitor's central axis.

$V = Ed$ and $V = \frac{V}{d}$.

$$\oint \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 \left(I_{\text{enclosed}} + \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \right)$$

$$B 2\pi r_{\text{path}} = \mu_0 \varepsilon_0 \pi r_{\text{flux}}^2 \frac{\mathrm{d}E}{\mathrm{d}t}$$

Then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{d}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{V_0 \omega}{d}\cos(\omega t)$$

and so

$$B = \frac{\mu_0 \varepsilon_0 r_{\text{flux}}^2}{2 r_{\text{path}}} \left(\frac{\omega V_0}{d} \right) \cos(\omega t)$$